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The Casual Characteristics of Offset Curves of a Non-Null Curve and Fibonacci Sequence

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Abstract. Fibonacci sequence is a very interesting sequence. In this paper, first the casual characteristics of the involute curve, and Mannheim partner of a non-null curve are examined. We find that the number of different forms of the casual characteristics of n^{th} order involute and Mannheim mate of a spacelike curve can be given using Fibonacci sequence with in a table. Also the kind of the casual characteristics of the n^{th} order Bertrand mate are examined. It is found that timelike Bertrand curve has 2^n different forms, and the n^{th} order Bertrand mate of a spacelike Bertrand curve with *timelike* principal normal is always spacelike Bertrand curve with timelike principal normal.

1. Introduction and Preliminaries

Bertrand mate, involute curve, Mannheim partner of a curve are well-known concepts in $\mathbf{E}^3$. Second order Mannheim partner and the second order involute curves in Euclidean 3-space are examined in [5] and [3], respectively. In this study, the casual characteristics of Bertrand mate α_B, involute curve α_I, Mannheim partner α_M of a non-null curve α are examined in 3-dimensional Lorentz space with notation IL^3. Also the kind of the casual characteristics of the n^{th} order Bertrand mate, involute curve, Mannheim partner of a non-null curve are given in a table with the number of different forms of casual characteristics. In 3-dimensional Lorentz space IL^3.is known Lorentz metric with index one, and $\left\{IR^3, \langle , \rangle\right\}$ is 3-dimensional Lorentz space with notation IL^3_1. For $X \in IL^3$; the casual characteristics of any vector X, are

i) if , $\langle X, X\rangle > 0$, X is spacelike vector
ii) if , $\langle X, X\rangle < 0$, X is timelike vector
iii) if , $\langle X, X\rangle = 0$, X is light-like or null vector.

Also $\|X\| = \sqrt{|\langle X, X\rangle|}$is the norm of vector X [11]. In 3-dimensional Lorentz space IL^3.

$$\langle X, Y\rangle = -x_1 y_2 + x_2 y_2 + x_3 y_3 \tag{1}$$

is known Lorentz metric with index one, and $\left\{IR^3, \langle , \rangle\right\}$ is 3-dimensional Lorentz space with notation IL^3.Vectorel product of X and Y is

$$X \Lambda Y = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1) \tag{2}$$

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Bertrand curve was discovered by J. Bertrand in 1850. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve called Bertrand mate. Bertrand curves have the following fundamental properties; which are given in more detail in [4], and [8]. Let α and α_B be the arclengthed curves with the parameters s and s_1, with the Frenet vector fields T, N, B and T_B, N_B, B_B, respectively, in $\mathbf{E}^3$. Two curves $\{\alpha, \alpha_B\}$ are called Bertrand pairs curves if they have common principal normal lines [2, 6, 8]. Also α_B is called Bertrand mate. If the curve α_B is Bertrand mate of α, then we may write that

$$\alpha_B(s) = \alpha(s) + \lambda N(s) \tag{3}$$

and $|\lambda|$ is the distance between the arclengthed curves α and α_B. Since $\{N, N_B\}$ are linear depended, so we have the equations $N = N_B$, and $\langle T_B, N\rangle = 0$. Also if the curve α_B is Bertrand mate α, then we have that $\langle T_B, T\rangle = \cos\theta = constant$. The Frenet vector fields T_B, N_B, B_B and T_{B2}, N_{B2}, B_{B2} which are belong to the curves $\alpha_B(s)$ and $\alpha_{B2}(s_{12})$ with the arcparametres s_1 and s_{12}, respectively. Two curves $\{\alpha_B, \alpha_B\}$ are called Bertrand pairs curves if they have common principal normal lines [2, 6, 8, 9]. So α_B is Bertrand mate of Bertrand mate of α. $\alpha_B(s) = \alpha(s) + \lambda N(s)$ .Also α_{B2} is called second order Bertrand mate of α, then we may write that

$$\begin{aligned} \alpha_{B2}(s) &= \alpha_B(s) + \lambda_1 N_B(s) && (4)\\ &= \alpha(s) + \lambda N(s) + \lambda_1 N_B(s) && (5) \end{aligned}$$

and $|\lambda_1|$ is the distance between the arclengthed curves α_B and α_{B2}. Third order Bertrand mate of α is Bertrand mate of second order Bertrand mate of α, also it can be written as

$$\alpha_{B3}\ (s) = \alpha(s) + \lambda(s)T(s) + \lambda_1(s)N(s) + \lambda_2 N_{B2} \tag{6}$$

since $\{N_B, N_{B2}\}$ are linear depended, so we have the equations $N_B = N_{B2}$ and $\langle T_{B2}, N_B\rangle = 0$. In similiar way nth order Bertrand mate of Bertrand curve α_B is n times Bertrand mate of Bertrand curve of α_B and written as

$$\alpha_{Bn}(s) = \alpha(s) + \lambda(s)N(s) + \lambda_1(s)N_B(s) + ... + \lambda_{n-1}N_{B(n-1)} \tag{7}$$

Involute-evolute curves are studied in differential geometry books in Euclidean 3-space. The involute of the curve is called sometimes the evolvent. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve [6]. In Lorentz space there are two kind of non-null curve, which are timelike and spacelike. Some characterizations for the pair of involute-evolute curves in [1]. Let $\alpha_I : I \to IL^3$ be unit speed curve with Frenet-Serret vectors $\{T_I, N_I, B_I\}$. If the tangent vector T of the curve α is lines to perpendicular on the tangent vector T_I of the curve α_2, hence if a curve α_2 is an involute of α and then it has the equation,

$$\alpha_I(s) = \alpha(s) + (c - s)T(s), \tag{8}$$

where $c = constant$. [2, 6, 7], that is Frenet vectors give us $N = T_I$ and $\langle T, T_I\rangle = 0$. $\alpha_2(s_2)$ is the involute of the curve $\alpha(s)$. Let $\alpha_{I2}(s_2)$ be the involute of the involute of curve $\alpha(s)$,also α_{I2} is called the second order involute curve α,

$$\begin{aligned} \alpha_{I2}(s) &= \alpha_I(s) + \lambda_{I2}T_{I2}(s) && (9)\\ &= \alpha(s) + (c - s)T(s) + (c_2 - s)N(s) && (10) \end{aligned}$$

is the parametrization of second order involute curve, since $\alpha_I(s) = \alpha(s) + (c - s)T(s)$. Third order involute curve of an evolute α is the involute curve of second order involute curve and can be written as

$$\begin{aligned} \alpha_{I3}(s) &= \alpha_{22}(s) + \lambda_{I2}T_{I2}\\ &= \alpha(s) + (c - s)T(s) + (c_2 - s)N(s) + \lambda_{I2}T_{I2} \end{aligned}$$

In similiar way n times involute of an evolute α_I is called n^{th} order involute of evolute α_I and can be writen as

$$\alpha_{In}(s) = \alpha(s) + (c-s)T(s) + (c_2 - s)N(s) + \lambda_{I2}T_{I2} + ... + \lambda_{I(n-1)}T_{I(n-1)}. \tag{11}$$

The Mannheim curve was first defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\frac{k_1}{(k_1^2+k_2^2)}$ is a nonzero constant, k_1 is the curvature and k_2 is the torsion. Recently, a new definition of the associated curves were given by Liu and Wang [10]. Mannheim curve was redefined by Liu and Wang. According to this new definition, if the the principal normal vector of the first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called the Mannheim partner curve. As a result, they called these new curves as Mannheim partner curves. For more detail see in [10]. Let $\alpha_M : I \to E^3$ be the $C^2 - class$ differentiable unit speed with $\{T_M, N_M, B_M\}$ be the Frenet frames. If the principal normal vector N of the curve α is linearly dependent on the binormal vector B_M of the curve α_M, then the pair $\{\alpha, \alpha_M\}$ is said to be Mannheim pair, then α is called a Mannheim curve and α_M is called Mannheim partner curve of α. Mannheim partner curve of α can be represented $\alpha = \alpha_M + \lambda B_M$. for some function λ_M, since N and B_M are linearly dependent, the equation can be rewritten as

$$\alpha_M(s) = \alpha(s) - \lambda_M B_M(s). \tag{12}$$

Also $N = B_M$, $\langle B_M, T\rangle = 0$, $< (T, T_M) = \cos\theta$ and besides the equality
$\lambda_M = \frac{k_1}{k_1^2+k_2^2}$ = constant is known the offset property, for some non-zero constant, for more detail, see in.[12].

Let $\{\alpha, \alpha_M\}$ and $\{\alpha_M, \alpha_{M2}\}$ be the Mannheim pairs of α and α_M repectively. We called as α_{M2} is a *Second order Mannheim partner* of the curve α, which has the following parametrization,

$$\alpha_{M2} = \alpha + \lambda_M sin\theta T - \lambda N + \lambda_M cos\theta B \tag{13}$$

since $\alpha_M(s) = \alpha(s) - \lambda_M B_M(s)$. Let $\{\alpha, \alpha_M\}$ and $\{\alpha_M, \alpha_{M2}\}$ and $\{\alpha_{M2}, \alpha_{M3}\}$ be the Mannheim pairs of α_M , α_{M2} and α_{M3} repectively. We called as α_{M3} is a third order Mannheim partner of the curve α. which has the following parametrizations, *third order Mannheim partner* α_{M3} can be written as

$$\begin{aligned} \alpha_{M3}(s) &= \alpha_{M2} - \lambda_{M2}B_{M2} \\ &= \alpha + \lambda_M sin\theta T - \lambda N + \lambda_M cos\theta B - \lambda_{M2}B_{M2}. \end{aligned}$$

In similiar way; n times Mannheim partner of Mannheim curve α_M is called, n^{th} order Mannheim partner of Mannheim curve α_M.

2. The casual characteristics of offset curves of a non-null curve and Fibonacci sequence

First, the casual characteristics of higher order involute of a non null curve will be examined in IL^3. The casual characteristics of higher order involute of a non null curve in IL^3 can be given as in the following table.

Theorem 2.1. *The casual characteristics of higher order involute of a spacelike curve with timelike normal, can be* *given as in the following table, in IL^3;*

evolute	**involute**	**2^{nd}involute**	**3^{rd}involute**	**4^{th}involute**	**5^{th}involute**	**6^{th}involute**	
$1\ diff$	$1\ diff$	$2\ diff$	$3\ diff$	$5\ diff$	$8\ diff$	$5+8\ diff$	
sts	$\rightarrow tss$	sst sts	$<$ sst sts $\rightarrow tss$	sst $<$ sts $\rightarrow tss$ sst $<$ sts	$<$ sst sts $\rightarrow tss$ $<$ sst sts $<$ sst sts $\rightarrow tss$	sst $<$ sts $\rightarrow tss$ $<$ sst sts $\rightarrow$ sst sts $\rightarrow tss$ sst $<$ sts $\rightarrow tss$ sst sts	.. ...

Proof. For a *spacelike* evolute curve with *timelike* normal N, binormal B is *spacelike*.

$$\left\{\begin{array}{ccc} T\ spacelike & N\ timelike & B\ spacelike \\ s & t & s \end{array}\right\} \tag{14}$$

Since $<T,T_I>=0$ and $T_I = N$ (*timelike*), it is trivial that T_I must be *timelike*. The involute of a spacelike curve with *timelike* normal is always timelike curve.

$$\left\{\begin{array}{ccc} T_I\ timelike & N_I\ spacelike & B_I\ spacelike \\ t & s & s \end{array}\right\} \tag{15}$$

Hence a *spacelike* evolute curve with *timelike* normal N, has the casual characteristics as in the following form

$$\begin{array}{cc} \textbf{evolute} & \textbf{involute} \\ \textbf{sts} \rightarrow & \textbf{tss} \end{array} \tag{16}$$

For a *spacelike* evolute curve with *timelike* binormal, principal normal N is *spacelike*, hence

$$\left\{\begin{array}{ccc} T\ spacelike & N\ spacelike & B\ timelike \\ s & s & t \end{array}\right\} \tag{17}$$

Since $<T,T_I>=0$ and $T_I = N$ (*spacelike*), tangent T_I must be *spacelike*.Hence the involute of a spacelike curve with *timelike* binormal is always spacelike curve. So normal N_I and binormal B_I must be *spacelike*

$$\left\{\begin{array}{ccc} Tangent\ T_I & Normal\ N_I & Binormal\ B_I \\ s & s & t \\ s & t & s \end{array}\right\} \tag{18}$$

The casual characteristics of *spacelike* evolute and *spacelike* involute have the following two forms

$$\begin{array}{cc} \textbf{evolute} & \textbf{involute} \\ \textbf{sst} & < \begin{array}{c} \textbf{sst} \\ \textbf{sts} \end{array} \end{array} \tag{19}$$

For a *timelike* evolute curve with *timelike* tangent vector T, normal vector N and binormal vector B are *spacelike*, lets use the following table

$$\left\{\begin{array}{ccc} T\ timelike & N\ spacelike & B\ spacelike \\ t & s & s \end{array}\right\} \tag{20}$$

for evolute-involute curve we know that $N = T_I$ and $< T, T_I >= 0$. Since tangent vector T is timelike and two timelike vectors are never orthogonal, T_I is not timelike, hence it is trivial that a timeline curve has always spacelike involute curve. Further a timelike evolute curve has spacelike involute curve with timelike normal N_I, or timelike normal B_I.

$$\left\{\begin{array}{ccc} Tangent\ T_I & Normal\ N_I & Binormal\ B_I \\ s & s & t \\ s & t & s \end{array}\right\} \tag{21}$$

The casual characteristics of timelike evolute and spacelike involute have the following two forms

$$\begin{array}{cc} \textbf{evolute} & \textbf{involute} \\ \textbf{tss} & < \begin{array}{c}\textbf{sst}\\ \textbf{sts}\end{array} \end{array} \tag{22}$$

The casual characteristics of Second order involute curve of *spacelike* evolute with timelike normal N, is a spacelike curve with *timelike* binormal, or *spacelike* binormal. If we use the form tables 16, 19 and 22 we have the following

$$\begin{array}{ccc} \textbf{evolute} & \textbf{involute} & \mathbf{2^{nd}involute} \\ & \textbf{1 different} & \textbf{2 different} \\ \textbf{sts} & \textbf{tss} & < \begin{array}{c}\textbf{sst}\\ \textbf{sts}\end{array} \end{array} \tag{23}$$

It is trivial from the 16, 19 , 22 and 23. If we go on we have the table with casual characteristics of higher order involute of a spacelike curve with timelike normal. □

Corollary 2.2. *The casual characteristics of higher order involute of a timelike curve has the following table*

$$\begin{array}{cccccc} \textbf{evolute} & \textbf{involute} & \mathbf{2^{nd}involute} & \mathbf{3^{rd}involute} & \mathbf{4^{th}involute} & \mathbf{5^{th}involute} \\ 1\ diff & 2\ diff & 3\ diff & 5\ diff & 8\ diff & 5+8\ diff \\ \to tss & \begin{array}{c} sst \\ sts \end{array} & \begin{array}{c} < \begin{array}{c} sst \\ sts \end{array} \\ \to tss \end{array} & \begin{array}{c} < \begin{array}{c} sst \\ sts \end{array} \\ \to tss \\ < \begin{array}{c} sst \\ sts \end{array} \end{array} & \begin{array}{c} < \begin{array}{c} sst \\ sts \end{array} \\ \to tss \\ < \begin{array}{c} sst \\ sts \end{array} \\ < \begin{array}{c} sst \\ sts \end{array} \\ \to tss \end{array} & \begin{array}{c} sst \\ < \\ sts \\ \to tss \\ \to \begin{array}{c} sst \\ sts \end{array} \\ \to tss \\ sst \\ < \\ sts \\ \to tss \\ sst \\ sts \\ < \begin{array}{c} sst \\ sts \end{array} \end{array} & \cdots \end{array}$$

Definition 2.3. *In mathematics, the Fibonacci numbers are the numbers in the following integer sequence, called the Fibonacci sequence, and characterized by the fact that every number after the first two is the sum of the two preceding ones:*

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ ...\} \tag{24}$$

Theorem 2.4. *The number of the different type the casual characteristics of n^{th} order involute of a spacelike curve timelike normal can be given using Fibonacci numbers.*

Proof. *In each step for tss and sst, there are two options, but for sts there is only one form so we can write the numbers of the forms by one by as in the following way*

$$\begin{aligned}
&& 1\\
&& 1\\
2 &=& 2.1 + 0.1\\
2 + 1 &=& 2.1 + 1.1\\
2 + 1 + 2 &=& 2.2 + 1.1\\
2 + 1 + 2 + 2 + 1 &=& 2.3 + 2.1\\
2 + 1 + 2 + 2 + 2 + 2 + 1 &=& 2.5 + 2.1
\end{aligned}$$

so we can write the following sequence

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 1 = 2.0 + 1.1\\
a_2 &= 2 = 2.1 + 0.1\\
a_3 &= 3 = 2.1 + 1.1 = 2.a_1 + a_0.1\\
a_4 &= 5 = 2.2 + 1.1 = 2.a_2 + a_1.1\\
a_5 &= 8 = 2.3 + 2.1 = 2.a_3 + a_2.1\\
a_6 &= 13 = 2.5 + 3.1 = 2.a_4 + a_3.1\\
a_7 &= 21 = 2.8 + 5.1 = 2.a_5 + a_4.1
\end{aligned}$$

$$a_n = 2.a_{n-2} + a_{n-3}.1$$

with so its general term is

$$a_n = 2.a_{n-2} + a_{n-3}.1, \quad n \geq 3. \tag{25}$$

Hence

$$a_{n+1} = 2.a_{n-1} + a_{n-2}.1 \tag{26}$$

and if we write and add the first two terms

$$\begin{aligned}
a_n &= 2.a_{n-2} + a_{n-3}.1,\\
a_{n-1} &= 2.a_{n-3} + a_{n-4}.1
\end{aligned}$$

we find

$$\begin{aligned}
a_n + a_{n-1} &= 2.a_{n-2} + a_{n-3}.1 + 2.a_{n-3} + a_{n-4}.1\\
&= 2.a_{n-2} + 2.a_{n-3} + a_{n-3}.1 + a_{n-4}.1\\
&= 2\left(a_{n-2} + a_{n-3}\right) + \left(a_{n-3} + a_{n-4}\right)\\
&= 2a_{n-1} + a_{n-2}\\
&= a_{n+1}.
\end{aligned}$$

When we calculate the number of the different forms we find the Fibonacci numbers. This complete the proof. □

The casual characteristics of higher order *Mannheim partner* of a non null *Mannheim* curve in IL^3 can be given as in the following table. For a non-null Mannheim curve there are the following forms;

Theorem 2.5. *The the casual characteristics of n^{th} order* Mannheim partner *of a spacelike curve with timelike normal can be given as in the following table, in IL^3.*

Mann. curve	**Mann. part.**	**2^{nd} Mann. part.**	**3^{rd} Mann. part.**	**4^{th} Mann. part.**	**5^{th} Mann. part.**	**6^{th} Mann. part.**	
$1\ dif$	$1\ dif$	$2\ dif$	$3\ dif$	$2+3\ dif$	$3+5\ diff$	$5+8\ dif$	
$sts \rightarrow$	$sst <$	$tss <$	$tss <$	$tss <$	$tss <$	$tss <$	... (27)
		$sts \rightarrow$	$sts \rightarrow$	$sts \rightarrow$	$sts \rightarrow$	$sts \rightarrow$	
			$sst <$	$sst <$	$sst <$	$sst <$	
				$tss <$	$tss <$	$tss <$	
				$sts \rightarrow$	$sts \rightarrow$	$sts \rightarrow$	
					$tss <$	$tss <$	
					$sts \rightarrow$	$sts \rightarrow$	
					$sst <$	$sst <$	
						$tss <$	
						$sts \rightarrow$	
						$sst <$	
						$tss <$	
						$sts \rightarrow$	

Proof. For a *spacelike* Mannheim curve with timelike normal N, binormal B is *spacelike* , that is

$$\left\{ \begin{array}{ccc} T\ spacelike & N\ timelike & B\ spacelike \\ s & t & s \end{array} \right\} \tag{28}$$

since $N = B_M$ (*timelike*) *and* $< N, T_M >= 0$, hence B_M must be *timelike*. Hence it is trivial that tangent T_M and normal N_M must be *spacelike*

$$\left\{ \begin{array}{ccc} T_M\ spacelike & N_M\ spacelike & B_M\ timelike \\ s & s & t \end{array} \right\} \tag{29}$$

hence we have

$$\begin{array}{cc} \textbf{Mann. curve} & \textbf{Mann. partner} \\ \textbf{sts} & \textbf{sst} \end{array} \tag{30}$$

For a *spacelike* Mannheim curve with timelike binormal B, normal N and tangent T are *spacelike* ,

$$\left\{ \begin{array}{ccc} T\ spacelike & N\ timelike & B\ spacelike \\ s & s & t \end{array} \right\} \tag{31}$$

since $N = B_M$ (*spacelike*) and $< N, T_M >= 0$, hence B_M must be *spacelike*. Hence it is trivial that tangent T_M or normal N_M must be *spacelike*, that is; it is trivial that; a *spacelike* Mannheim curve with timelike binormal B has Mannheim partner with always spacelike binormal vector B_M. Mannheim partner is *timelike* or *spacelike*.

$$\begin{array}{ccc} Tangent\ T_M & Normal\ N_M & Binormal\ B_M \\ s & t & s \\ t & s & s \end{array}. \tag{32}$$

Hence the casual characteristics of have the following two forms

$$\begin{array}{ccc} \textbf{Mann. curve} & & \textbf{Mann. partner} \\ \textbf{sst} & < & \begin{array}{c}\textbf{sts}\\ \textbf{tss}\end{array} \end{array} \tag{33}$$

For a *timelike* Mannheim curve with *timelike* tangent vector T, normal vector N and binormal vector B are *spacelike*, lets use the following table

$$\left\{ \begin{array}{ccc} T\ timelike & N\ spacelike & B\ spacelike \\ t & s & s \end{array} \right\} \tag{34}$$

We know that for *Mannheim* pairs $N = B_M$ (*spacelike*) and $< N, T_M >= 0$, hence it is trivial that; a timelike curve has spacelike or time like Mannheim partner with always spacelike binormal vector B_M,

$$\left\{ \begin{array}{ccc} Tangent\ T_M & Normal\ N_M & Binormal\ B_M \\ s & t & s \\ t & s & s \end{array} \right\} \tag{35}$$

The casual characteristics of *timelike* Mannheim curve and Mannheim partner have the following two forms

$$\begin{array}{ccc} \textbf{Mann.curve} & & \textbf{Mann.partner} \\ \textbf{tss} & < & \begin{array}{c}\textbf{tss}\\ \textbf{sts}\end{array} \end{array} \tag{36}$$

The casual characteristics of Second order Mannheim partner of *spacelike* Mannheim curve with timelike normal N, is a spacelike curve with *timelike* binormal or *spacelike* binormal. It is trivial that

$$\begin{array}{ccc} Mann.\ curve & Mann.part & 2^{nd}\ order\ Mann.part \\ sts \rightarrow & sst < & \begin{array}{c} tss \\ sts \end{array} \end{array} \tag{37}$$

from the 30, 33 and 36. If we go on, we have the table with casual characteristics of higher order Mannheim partner of a spacelike curve with *timelike* normal, as a result, we have the proof. □

Corollary 2.6. *The casual characteristics of higher order Mannheim partner of a timelike Mannheim curve has the following table*

$$\begin{array}{cccccc} \textbf{Mann.} & \textbf{Mann.} & 2^{nd}\ \textbf{Mann.} & 3^{th}\ \textbf{Mann.} & 4^{th}\ \textbf{Mann.} & \\ \textbf{curve} & \textbf{part.} & \textbf{part.} & \textbf{part.} & \textbf{part.} & \\ & 2\ dif. & 3\ dif. & 5\ dif. & 8\ dif. & \\ tss < & \begin{array}{c} tss < \\ sts \rightarrow \end{array} & \begin{array}{c} tss < \\ sts \rightarrow \\ sst < \end{array} & \begin{array}{c} tss < \\ sts \rightarrow \\ sst < \\ tss < \\ sts \rightarrow \end{array} & \begin{array}{c} tss < \\ sts \rightarrow \\ sst < \\ tss < \\ sts \rightarrow \\ tss < \\ sts \rightarrow \\ sst < \end{array} & \ldots \end{array} \tag{38}$$

Theorem 2.7. *The number of the different type the casual characteristics of n th order* Mannheim partner *of a spacelike curve with timelike normal can be given using Fibonacci numbers.*

Proof. In each step for forms **sts** *and* **sst**, *there are two options, but for the form tss there is only one form so we can write the numbers of the forms by one by as similiar in the theorem 2.2 using Fibonacci numbers.* □

2.1. The casual characteristics of higher order Bertrand mate

The casual characteristics of higher order Bertrand mate of a non-null Bertrand curve in IL^3 can be given as in the following table. For a non-null Bertrand mate of non-null Bertrand curves there are the following forms;

Theorem 2.8. *The number of the n^{th} order Bertrand mate with different type of casual characteristics of a spacelike Bertrand curve with timelike binormal and a timelike Bertrand curve is given as 2^n, also the n^{th} order Bertrand mate of a spacelike Bertrand curve with timelike principal normal is always spacelike curve with timelike principal normal.*

timelike Bert. curve	**Bert. mate**	**2^{nd} Bert. mate**	**3^{rd}Bert. mate**	**4^{th}Bert. mate**	..	**n^{th}Bert. mate**
$1\ different$	$2\ different$ $= 2^2$	$4\ different$ $= 2^2$	$8\ different$ $= 2^3$	16 $different$	..	$2^n different$
$tss <$	sst tss	$< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$	$< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$	$< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$ $< \begin{matrix} sst \\ tss \end{matrix}$		

spacelike Bert. curve with timelike normal	*Bert mate*	*2^{nd} Bert mate*	*3^{rd}Bert mate*	*4^{th}Bert mate*	*n^{th}Ber mate*
sts	$\to sts$	$\to sts$	$\to sts$	$\to sts$	$\to sts$

(39)

Proof. If α isa timelike Bertrand curve, it has *timelike* tangent vector T,and normal N and binormal B are *spacelike* .

$$\left\{ \begin{matrix} T\ timelike & N\ spacelike & B\ spacelike \\ t & s & s \end{matrix} \right\} \tag{40}$$

Also since $N = N_B$, N_B is *spacelike* hence normal N_B and binormal B_B can be *timelike* or *spacelike*. and $N = N_B$, $\langle T_B, N \rangle = 0$. Hence

$$\left\{ \begin{matrix} Tangent\ T_B & Normal\ N_B & Binormal\ B_B \\ s & s & t \\ t & s & s \end{matrix} \right\} \tag{41}$$

are the forms of the casual characteristics of Bertrand mate. The Bertrand mate of a *timelike* Bertrand curve is either spacelike curve with timelike binormal or *timelike* curve, and we will show this result as in the following way;

Bertrand	**Bertrand mate**
tss	$< \begin{matrix} sst \\ tss \end{matrix}$

(42)

For a *spacelike* Bertrand curve with *timelike* binormal vector as in

$$\left\{ \begin{array}{ccc} T\ spacelike & N\ spacelike & B\ timelike \\ s & s & t \end{array} \right\} \tag{43}$$

it is trivial that a *spacelike* curve has *spacelike* tangent vector T , with timelike binormal B and is *spacelike* normal N, since $N = N_B$, normal N_B is always *spacelike*,and

$$N = N_B, \qquad \langle T_B, N \rangle = 0 \tag{44}$$

hence and *Binormal* can be *timelike* or *spacelike*, it is trivial

$$\left\{ \begin{array}{ccc} Tangent\ T_B & Normal\ N_B & Binormal\ B_B \\ s & s & t \\ t & s & s \end{array} \right\} \tag{45}$$

Hence the Bertrand mate of a *spacelike* Bertrand curve with *timelike* binormal vector B, is either *timelike* curve or *spacelike* curve with *timelike* binormal. They have following casual charastics;

$$\begin{array}{cc} \textbf{Bertrand} & \textbf{Bertrand mate} \\ \textbf{sst} & < \begin{array}{c} sst \\ tss \end{array} \end{array} \tag{46}$$

For a *spacelike* Bertrand curve with *timelike* normal as in

$$\left\{ \begin{array}{ccc} T\ spacelike & N\ timelike & B\ spacelike \\ s & t & s \end{array} \right\} \tag{47}$$

Bertrand mate of a *spacelike* Bertrand curve with *timelike* normal N curve since and $N = N_B$, $\langle T_B, N \rangle = 0$ and Normal N_B is always *timelike*, also T_B is *spacelike*, hence Bertrand mate is *spacelike* , we have

$$\left\{ \begin{array}{ccc} T_B\ spacelike & N_B\ timelike & B_B\ spacelike \\ s & t & s \end{array} \right\} \tag{48}$$

Bertrand mate of a *spacelike* Bertrand curve with *timelike* normal vector is always *spacelike* curve with *timelike* normal as in following casual charastics;

$$\begin{array}{cc} \textbf{Bertrand curve} & \textbf{Bertrand mate} \\ \textbf{sts} & \rightarrow \textbf{sts} \end{array} \tag{49}$$

Using the 42, 46 and 49, we have the following table

$$\begin{array}{cc} \textbf{timelike Bert. curve} & \textbf{Bert. mate} \\ tss & < \begin{array}{c} sst \\ sts \end{array} \\ \begin{array}{c} \textbf{spacelike Bert. curve} \\ \textbf{timelike binormal} \\ sst \end{array} & < \begin{array}{c} sst \\ sts \end{array} \\ \begin{array}{c} \textbf{spacelike Bert. curve} \\ \textbf{timelike normal} \end{array} & Bert.\ mate \\ sts & sts \end{array} \tag{50}$$

The casual characteristics of second order *Bertrand mate* of a *non − null* Bertrand curve is trivial from the 42, 46 and 49

$$\begin{array}{ccc}
\textbf{timelike} & \textbf{Bert. mate} & \mathbf{2^{nd}}\textbf{ Bert. mate} \\
\textbf{Bert. curve} & \textbf{2 different} & \textbf{4 different} \\
tss < & \begin{array}{c} sst \\ sts \end{array} & < \begin{array}{l} sst < \begin{array}{c} ss \\ tss \end{array} \\ tss < \begin{array}{c} sst \\ tss \end{array} \end{array} \\
\begin{array}{c} \textbf{spacelike} \\ \textbf{Bert. curve} \\ \textbf{timelike binormal} \\ \textbf{sst} \end{array} & \begin{array}{c} sst \\ sts \end{array} & < \begin{array}{l} sst < \begin{array}{c} sst \\ tss \end{array} \\ tss < \begin{array}{c} sst \\ tss \end{array} \end{array} \\
\begin{array}{c} \textbf{spacelike} \\ \textbf{Bert. curve} \\ \textbf{timelike normal} \\ \textbf{sts} \end{array} & \begin{array}{c} \textbf{1 Bert. mate} \\ \\ \textbf{sts} \end{array} & \begin{array}{c} \textbf{1 Bert. mate} \\ \\ \textbf{sts} \end{array}
\end{array} \tag{51}$$

□

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