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ON q-BERNOULLI INEQUALITY

q-BERNOULLI EŞİTSİZLİĞİ ÜZERİNE

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Abstract. In this work, the q–analogue of Bernoulli inequality is proved. Some other related results are presented.

Keywords q-Bernoulli inequality, q-Calculus, Combinatorial inequalities

1. Introduction

Throughout this work, we consider $q \in (0,1)$. The q-number is defined to be the number of the form

$$[\alpha]_q = \frac{q^\alpha - 1}{q-1}, \qquad \text{for any } \alpha \in \mathbb{C}.$$

In particular, if $\alpha = n \in \mathbb{N}$, then the positive q-integer is defined to be

$$[n]_q = \frac{q^n - 1}{q-1} = 1 + q + q^2 + \cdots q^{n-1}.$$

In special case, we have $[1]_q = 1$ and $[0]_q = \frac{1}{1-q} = [\infty]_q$.

We define the q-factorial of the number $[n]_q$ and the g-binomial coefficient by

$$[0]_q! = 1, \qquad [n]_q! = [n]_q [n-1]_q \cdots [2]_q \cdot [1]_q \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q! \, [n-j]_q!}$$

with the convention that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1, \qquad \begin{bmatrix} 0 \\ j \end{bmatrix}_q = 0, \ \forall j \geq 1.$$

The q-Pochammer symbol is defined to be

$$(x-a)_q^n = \prod_{j=0}^{n-1} \left(x - q^j a\right), \ \text{ with } \ (x-a)_q^{(0)} = 1, \ \text{ and } \ (x-a)_q^{-n} = \frac{1}{(x - q^{-n}a)_q^n}.$$

This formula plays an important role in combinatorics. For instance, for $x = 1$ and $x = a$ this formula make sense as $n = \infty$:

$$(1+x)_q^n = \prod_{j=0}^{\infty} \left(1 + q^j x\right).$$

The above infinite product converges if $q \in (0, \infty)$.

We adopt the symbol

$$(1+x)_q^{\alpha} = \frac{(1+x)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}}$$

for any number α. Clearly, this definition coincides with definition of $(1+x)_q^n$ when $\alpha = n \in \mathbb{N}$.

Lemma 1. [2] *For any two numbers α and β, we have*

$$(1+x)_q^{\alpha} = \frac{(1+x)_q^{\alpha+\beta}}{(1+q^{\alpha}x)_q^{\beta}}$$

and

$$D_q\,(1+x)_q^{\alpha} = [\alpha]_q\,(1+qx)_q^{\alpha-1}.$$

The q-derivative of any real valued function f is defined to be

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)\,x}, \qquad x \neq 0.$$

Clearly, as $q \to 1^-$ then $D_q f(x)$ tends to $f'(x)$, provided that f is differentiable.

Two fundamentals q-binomial formulas are well know in Literature. The q-Gauss binomial which has the form

$$(1+x)_q^n = \sum_{j=0}^{n} \left[\begin{array}{c} n \\ j \end{array}\right]_q q^{j(j-1)/2} x^j$$

and the q-Heine's binomial formula

$$\frac{1}{(1-x)_q^n} = \sum_{j=0}^{\infty} \left[\begin{array}{c} n \\ j \end{array}\right]_q x^j.$$

However, since

$$\lim_{n\to\infty} \left[\begin{array}{c} n \\ j \end{array}\right]_q = \frac{1}{(1-q)(1-q^2)\dots(1-q^j)}$$

Applying this for q-Gauss and q-Heine's binomial formulas, we get two formal power series in x. Namely, we have

$$(1+x)_q^{\infty} = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)}, \tag{1.1}$$

and

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{j=0}^{\infty} \frac{x^j}{(1-q)(1-q^2)\dots(1-q^j)}. \tag{1.2}$$

These two series are very useful in the theory of q-calculus, since they were used to define the q-analogue of exponential function. From (1.2)

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{j=0}^{n} \frac{\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\cdots\left(\frac{1-q^j}{1-q}\right)} = \sum_{j=0}^{n} \frac{\left(\frac{x}{1-q}\right)^j}{[1]_1\,[2]_q\cdots[j]_q} = \sum_{j=0}^{n} \frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!} = \mathrm{e}_q^{\frac{x}{1-q}},$$

or we write

$$\mathrm{e}_q^x = \frac{1}{(1-(1-q)\,x)_q^{\infty}}.$$

Similarly, if we use (1.1) the companion q-exponential function is defined to be

$$(1+x)_q^{\infty} = \sum_{j=0}^{n} \frac{q^{j(j-1)/2}\left(\frac{x}{1-q}\right)^j}{\left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\cdots\left(\frac{1-q^j}{1-q}\right)} = \sum_{j=0}^{n} \frac{q^{j(j-1)/2}\left(\frac{x}{1-q}\right)^j}{[1]_1\,[2]_q\cdots[j]_q} = \sum_{j=0}^{n} \frac{\left(\frac{x}{1-q}\right)^j}{[j]_q!} = \mathrm{E}_q^{\frac{x}{1-q}}$$

or we write

$$\mathrm{E}_q^x = (1+(1-q)\,x)_q^{\infty}.$$

The derivatives of the above two q-exponential functions are given as

$$D_q\mathrm{E}_q^x = \mathrm{E}_q^{qx}, \qquad D_q\mathrm{e}_q^x = \mathrm{e}_q^x$$

We note that the additive property of the exponentials does not hold in general, i.e.,

$$\mathrm{e}_q^x\mathrm{e}_q^y = \mathrm{e}_q^{x+y}.$$

However, if x and y satisfy the commutation relation $yx = qxy$, then the additive property holds.

The two functions E_q^x and e_q^x are connected to each other by the relations

$$\mathrm{E}_q^{-x}\mathrm{e}_q^x = 1, \qquad \mathrm{e}_{1/q}^x = \mathrm{E}_q^x.$$

Naturally, it is important to know the relation between these q-quantities. One of the most effective method is to use inequalities. Among others, one of the most famous and applicable inequalities used in mathematics is the Bernoulli inequality, which is well known as:

$$(1+x)^n \geq 1+nx \tag{1.3}$$

for every $x > -1$ and every positive integer $n \geq 1$. This was extended to more general form such as [3]:

$$(1+x)^{\alpha} \geq 1+\alpha x, \qquad \alpha \geq 1 \tag{1.4}$$

and

$$(1+x)^{\alpha} \leq 1+\alpha x, \qquad 0 < \alpha < 1. \tag{1.5}$$

This inequality has important applications in proving some classical theorems in Analysis and Statistics. Due to its important role, in this work we prove the q-analogue of Bernoulli inequality and give some other related inequalities.

2. q-Bernoulli inequality

Let us begin with the following version of q-Bernoulli inequality for integers.

Theorem 1. *Let $q \in (0,1)$. If $x > -1$ then the q-Bernoulli inequality*

$$(1+x)_q^n \geq 1+[n]_q\,x, \tag{2.1}$$

is valid for every positive integer $n \geq 1$.

Proof. Our proof carries by Induction. Define the statement

$$\mathrm{P}(n): \qquad (1+x)_q^n \geq 1+[n]_q\, x. \tag{2.2}$$

Case I: If $x=0$, then the we get equality for all n and thus (2.1) holds.

Case II: If $-1<x<0$. Let $x=-y$, $0<y<1$, so that (2.2) becomes

$$\mathrm{P}(n): \qquad (1-y)_q^n \geq 1+[n]_q\, y.$$

For $n=1$, we have

$$\mathrm{P}(1): \qquad (1-y)_q^1 = (1-y) \geq 1-y = 1-[1]_q\, y.$$

Assume $\mathrm{P}(n)$ holds for $n=k$, i.e.,

$$\mathrm{P}(k): \qquad (1-y)_q^k \geq 1-[k]_q\, y \qquad \text{is true.}$$

We need to show that

$$\mathrm{P}(k+1): \qquad (1+y)_q^{k+1} \geq 1-[k+1]_q\, y$$

is true?.

$$\begin{aligned}
(1-y)_q^{k+1} &= (1-y)_q^k \left(1-q^k y\right) \\
&\geq \left(1-[k]_q\, y\right)\left(1-q^k y\right) \qquad \text{(follows by assumption (2.2) for } n=k) \\
&= 1-q^k y-[k]_q\, y+q^k\,[k]_q\, y^2 \\
&\geq 1-q^k y-[k]_q\, y \\
&\geq 1-y-[k]_q\, y \qquad \text{(since } 0>-q^k>-1) \\
&= 1-\left(1+[k]_q\right) y \\
&\geq 1-[k+1]_q\, y \qquad \text{(since } ([k+1]_q = q\,[k]_q+1 \leq [k]_q+1)
\end{aligned}$$

which means the statement $\mathrm{P}(k+1)$ is true and thus by Mathematical Induction hypothesis the inequality in (2.1) holds for every $n \in \mathbb{N}$ and $-1<x<0$.

Case III: If $x>0$. Then,

$$\mathrm{P}(1): \qquad (1+x)_q^1 = (1+x) \geq 1+x = 1+[1]_q\, x$$

Assume (2.2) holds for $n=k$, i.e.,

$$\mathrm{P}(k): \qquad (1+x)_q^k \geq 1+[k]_q\, x \qquad \text{is true.}$$

We need to show that

$$\mathrm{P}(k+1): \qquad (1+x)_q^{k+1} \geq 1+[k+1]_q\, x$$

is true?.

Starting with the left-hand side

$$\begin{aligned}
(1+x)_q^{k+1} &= (1+x)_q^k \left(1+q^k x\right) \\
&\geq \left(1+[k]_q\, x\right)\left(1+q^k x\right) && \text{(follows by assumption (2.2) for } n=k) \\
&= 1+[k]_q\, x + q^k x + q^k [k]_q\, x^2 \\
&\geq 1+\frac{1}{q} q\,[k]_q\, x + \frac{q}{q} q^k x \\
&= 1+\frac{1}{q}\left([k+1]_q - 1\right) x + \frac{1}{q}\left(q^{k+1}x\right) && \text{(since } [k+1]_q = q\,[k]_q + 1) \\
&= 1+\frac{1}{q}\left([k+1]_q + q^{k+1} - 1\right) x \\
&= 1+\frac{1}{q}\left([k+1]_q + (q-1)\,[k+1]_q\right) x && \text{(since } q^{k+1}-1 = (q-1)\,[k+1]_q) \\
&= 1+[k+1]_q\, x,
\end{aligned}$$

which means the statement $\mathrm{P}(k+1)$ is true and thus by Mathematical Induction hypothesis the inequality in (2.1) holds for every $n \in \mathbb{N}$ and $x > 0$. Combining all above three cases I, II and III the inequality (2.1) holds for all $n \geq 1$ and all $x > -1$. $\square$

Remark 1. *As $q \to 1$ in (2.1), then the q-Bernoulli inequality (2.1) reduces to the original version of Bernoulli inequality (1.3) for integer case.*

Remark 2. *For the case $-1 < x < 0$, we prefer to write (2.1) in the form*

$$(1-y)_q^n \geq 1-[n]_q\, y$$

for every $0 < y < 1$ and $n \geq 1$.

Corollary 1. *Let $q \in (0,1)$. If $x > -1$, then the generalization q-Bernoulli inequality*

$$(1+x)_q^{m+n} \geq \left(1+[m]_q\, x\right)(1+q^m x)_q^n\,,$$

is valid for every $m \in \mathbb{N}$ and $n \in \mathbb{Z}$.

Proof. The result is an immediate consequence of Theorem 1, by substituting $(1+x)_q^m = \frac{(1+x)_q^{m+n}}{(1+q^m x)_q^n}$ in (2.1). So that the result follows for every $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. $\square$

The following generalization of (2.1) is valid for any real number $\alpha \geq 0$.

Theorem 2. *Let $q \in (0,1)$. If $x \geq 0$ then the q-Bernoulli inequality*

$$(1+x)_q^\alpha \geq 1+[\alpha]_q\, x, \qquad \alpha \geq 1 \tag{2.3}$$

and

$$(1+x)_q^\alpha \leq 1+[\alpha]_q\, x, \qquad 0 < \alpha < 1 \tag{2.4}$$

is valid.

Proof. Let us recall that [1], for $0 < a < b$ (or $0 > a > b$), a function $f(x)$ is said to be q-increasing (respectively, q-decreasing) on $[a,b]$, if $f(qx) \leq f(x)$ (respectively, $f(qx) \geq f(x)$) whenever, $x \in [a,b]$ and $qx \in [a,b]$. As a direct consequence we have,

$f(x)$ is q-increasing (respectively, q-decreasing) on $[a,b]$ iff $D_q f(x) \geq 0$ (respectively, $D_q f(x) \leq 0$), whenever, $x \in [a,b]$ and $qx \in [a,b]$.

Let $f(x) = (1+x)_q^{\alpha} - [\alpha]_q x - 1$, $x \geq 0$. Since $(1+x)_q^{\alpha} = \frac{(1+x)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}}$, inserting qx instead of x and replace α by $\alpha - 1$ we get $(1+qx)_q^{\alpha-1} = \frac{(1+qx)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}}$. Therefore, we have

$$\begin{aligned} D_q f(x) &= [\alpha]_q (1+qx)_q^{\alpha-1} - [\alpha]_q \\ &= [\alpha]_q \frac{(1+qx)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}} - [\alpha]_q \\ &= [\alpha]_q \frac{\sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} q^j x^j}{\sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} q^{j\alpha} x^j} - [\alpha]_q \qquad \left(\text{with the convention } \prod_{k=1}^{0}\left(1-q^j\right) = 1\right) \\ &= [\alpha]_q \frac{1 + \sum_{j=1}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} q^j x^j}{1 + \sum_{j=1}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} q^{j\alpha} x^j} - [\alpha]_q \\ &= [\alpha]_q \frac{\sum_{j=1}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} \left(q^j - q^{j\alpha}\right) x^j}{1 + \sum_{j=1}^{\infty} \frac{q^{j(j-1)/2}}{\prod_{k=1}^{j}(1-q^k)} q^{j(\alpha-1)} x^j} \geq 0, \end{aligned}$$

since $q \in (0,1)$ and $\alpha \geq 1$ then $\left(q^j - q^{j\alpha}\right) > 0$, and this implies that $D_q f(x) \geq 0$ for all $x \geq 0$, which means that f is q-increasing and thus the inequality (2.3) is proved.

The inequality (2.4) is deduced from the above proof by noting that $\left(q^j - q^{j\alpha}\right) < 0$ for all $0 < \alpha < 1$. □

Remark 3. *Setting $\alpha = n \in \mathbb{N}$ in (2.3), then the inequality (2.3) reduces to the q-version of Bernoulli inequality (2.1) for integer case but for $x \geq 0$. Moreover, as $q \to 1$ (2.3) and (2.4) reduces to the classical versions (1.4) and (1.5); respectively.*

Testing the validity of (2.3) and (2.4) for $-1 < x < 0$ arbitrarily, we find that these inequalities can be extended but with additional restriction on $q \in (0,1)$, as given in the following result.

Theorem 3. *There exists $\widehat{q} \in (0,1)$ such that the inequalities (2.3) and (2.4) are hold for every $q \in (\widehat{q}, 1)$ and every $x > -1$.*

Proof. Firstly, we need to recall the q-Mean Value Theorem (q-MVT) given in [4], it states that: For a continuous function g defined on $[a,b]$ $(0 < a < b)$, there exist $\eta \in (a,b)$ and $\widehat{q} \in (0,1)$ such that

$$g(b) - g(a) = D_q g(\eta)(b-a) \tag{2.5}$$

for all $q \in (\widehat{q}, 1)$.

Case I. If $x \geq 0$. We consider the function $f(t) = (1+t)_q^{\alpha}$ defined for $t \geq 0$. Clearly f is continuous for $t \in [0, x] \subset [0, \infty)$, and $D_q f(c) = [\alpha]_q (1+qc)_q^{\alpha-1}$. Applying, (2.5) for $a = 0$ and $b = x$ then there exist $\eta \in (a, b)$ and $\widehat{q} \in (0, 1)$

$$(1+x)_q^{\alpha} - 1 = [\alpha]_q (1+q\eta)_q^{\alpha-1} (x-0) \geq [\alpha]_q x \qquad \forall q \in (\widehat{q}, 1).$$

This yields that

$$(1+x)_q^{\alpha} \geq 1 + [\alpha]_q x$$

$\forall q \in (\widehat{q}, 1)$, and this proves (2.3).

Case II. If $-1 < x < 0$. Let us write (2.3) as follows:

$$1 - [\alpha]_q x \leq (1-x)_q^{\alpha}. \tag{2.6}$$

Consider the function $f(t) = (1-t)_q^{\alpha}$ defined for $0 \leq t \leq 1$. Clearly f is continuous for $t \in [0, x] \subset [0, 1]$, and $D_q f(c) = [\alpha]_q (1+qc)_q^{\alpha-1}$. Applying, (2.5) for $a = 0$ and $b = x$ then there exist $\eta \in (0, x)$ and $\widehat{q} \in (0, 1)$

$$(1-x)_q^{\alpha} - 1 = -[\alpha]_q (1-q\eta)_q^{\alpha-1} (x-0) \geq -[\alpha]_q x \qquad \forall q \in (\widehat{q}, 1). \tag{2.7}$$

This yields that

$$(1-x)_q^{\alpha} \geq 1 - [\alpha]_q x$$

$\forall q \in (\widehat{q}, 1)$ with $-1 < x < 0$, and this proves the inequality. The reverse inequality in (2.6) holds since the inequality in (2.7) is reversed for $0 < \alpha < 1$, which proves (2.4) $\square$

A generalization of (2.3) and (2.4) is given as follows:

Proposition 1. *Let $\beta \in \mathbb{R}$. There exists $\widehat{q} \in (0, 1)$ such that for every $x > -1$ the inequalities*

$$(1+x)_q^{\alpha+\beta} \geq \Big(1 + [\alpha]_q x\Big) (1+q^{\alpha}x)_q^{\beta} \qquad \alpha \geq 1 \tag{2.8}$$

and

$$(1+x)_q^{\alpha+\beta} \leq \Big(1 + [\alpha]_q x\Big) (1+q^{\alpha}x)_q^{\beta} \qquad 0 < \alpha \leq 1 \tag{2.9}$$

are hold for every $q \in (\widehat{q}, 1)$.

Proof. From Lemma 1 we have $(1+x)_q^{\alpha} = \frac{(1+x)_q^{\alpha+\beta}}{(1+q^{\alpha}x)_q^{\beta}}$. Substituting in (2.3) we get the required result. $\square$

Remark 4. *Setting $\beta = 0$ in (2.8) and (2.9) we recapture (2.3) and (2.4), respectively.*

Corollary 2. *Let $\beta \in \mathbb{R}$. There exists $\widehat{q} \in (0, 1)$ such that for every $x > -1$ the inequalities*

$$(1+x)_q^{\infty} \geq \Big(1 + [\alpha]_q x\Big) (1+q^{\alpha}x)_q^{\beta} \left(1+q^{\alpha+\beta}x\right)_q^{\infty} \qquad \alpha \geq 1 \tag{2.10}$$

and

$$(1+x)_q^{\infty} \leq \Big(1 + [\alpha]_q x\Big) (1+q^{\alpha}x)_q^{\beta} \left(1+q^{\alpha+\beta}x\right)_q^{\infty} \qquad 0 < \alpha \leq 1 \tag{2.11}$$

are hold for every $q \in (\widehat{q}, 1)$

Proof. Substituting $(1+x)_q^{\alpha+\beta} = \frac{(1+x)_q^\infty}{(1+q^{\alpha+\beta}x)_q^\infty}$ in (2.8) and (2.9); respectively, we get the required result. □

Remark 5. *Replacing* '$(1-q)x$' *instead of* 'x' *in* (2.10) *and* (2.11), *we get inequalities for the exponential function* E^x *for all* $x > \frac{-1}{1-q}$. *Similarly, for* e^x *wit a bit changes in the substitution.*

Corollary 3. *There exists* $\widehat{q} \in (0, 1)$ *such that for every* $x > -1$ *the inequalities*

$$(1+x)_q^\infty \geq \left(1 + [\alpha]_q\, x\right) (1+q^\alpha x)_q^\infty \qquad \alpha \geq 1 \tag{2.12}$$

and

$$(1+x)_q^\infty \leq \left(1 + [\alpha]_q\, x\right) (1+q^\alpha x)_q^\infty \qquad 0 < \alpha \leq 1 \tag{2.13}$$

are hold for every $q \in (\widehat{q}, 1)$

Proof. Setting $\beta = 0$ in (2.10) and (2.11); respectively, we get the required result. □

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