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On Countability Properties of Function Spaces with the $\mathbb{R}$-Compact-Open Topology

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Abstract. The purpose of this article is to examine the countability properties of the $\mathbb{R}$-compact-open topology on $RC(X)$ of all real-valued functions on X that are continuous on C-compact subsets of X, such as second countability, separability, and the properties of $\aleph_0$-space, Polish space and cosmic space.

1. Introduction and Preliminaries

Compact-open topology has an important place in the studies on function spaces and has many applications in homotopy theory and functional analysis. This topology emerged in 1945 in an article by Fox [1] and soon after was developed by Arens in [2] and by Arens and Dugundji in [3]. Since it is used to study uniformly convergent sequences of functions on compact subsets in [4], it is also called the topology of uniform convergence on compact sets. Based on this, many topologies have been defined between compact-open topology and uniform convergence topology. One of them is the $\mathbb{R}$-compact-open topology on $C(X)$ [5]. In [5] and [6], many topological properties of $C(X)$ space, such as metrizability, complete metrizability, and countability, are investigated. In recent years, many studies have been on topologies in function spaces (see [7–10]).

In the present study, we introduce $\mathbb{R}$-compact-open topology on $RC(X)$ of all real-valued functions on X that are continuous on C-compact subsets of X. The countability properties of the $\mathbb{R}$-compact-open topology on $RC(X)$, such as second countability, separability, and the properties of $\aleph_0$-space, Polish space and cosmic space are investigated.

Throughout this paper, unless otherwise stated clearly, all spaces are assumed to be Tychonoff (completely regular Hausdorff), and topological space will be used as space.

$\mathbb{R}$ denotes the real line with the natural topology. The topology of the space X will be represented by $\tau(X)$. The closure of a subset A in X is denoted by $\bar{A}$. If $A \subseteq X$, the restriction of a function $f \in C(X)$ to the set A is denoted by $f|_A$. Finally, if X and Y are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is ner than or equal to the topology on X, and that the topology on Y is strictly ner than the topology on X.

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2. The $\mathbb{R}$-compact-open topology on $RC(X)$

In this section, we remind the $\mathbb{R}$-compact-open topology on $RC(X)$ and also give an equivalent definition.

Recall that a subset A of a space X is called C-compact (or $\mathbb{R}$-compact) if, for any real-valued function f continuous on X, the set $f(A)$ is compact in $\mathbb{R}$. Note that in the case of $A = X$, the property of the set A to be C-compact coincides with the pseudocompactness of the space X.

A space X is a rc_f-space if a function $f : X \to \mathbb{R}$ is continuous if and only if for every C-compact subset $A \subseteq X$

the restriction $f|_A$ is continuous [6]. $\mathbb{R}^X$ denotes the set of all real-valued functions on space X. Let $RC(X) = \{f \in \mathbb{R}^X : f|_A$ is continuous for each C-compact subset A of $X\}$. Since the restriction function of every continuous function is also continuous, $C(X) \subseteq RC(X)$. Therefore, it is seen that $RC(X) = C(X)$ if and only if X is a rc_f-space. Also, if X is a submetrizable space, then it is rc_f-space. It is clearly seen that for C-compact space $RC(X) = C(X)$.

Let α be a nonempty family of all C-compact subsets of X. For $A \in \alpha$ and $V \in \tau(\mathbb{R})$, $\mathbb{R}$-compact-open topology on $RC(X)$ [6] denoted by $RC_{rc}(X)$ has a subbase consisting of the sets

$$S(A, V) = \{f \in RC(X) : f(A) \subseteq V\}.$$

Also this topology has as base at each point $f \in RC(X)$ the family of all sets of the form

$$B_A(f, \epsilon) = \{g \in RC(X) : \sup_{x \in A} \left| f(x) - g(x) \right| < \epsilon\}$$

where $A \in \alpha$ and $\epsilon > 0$.

Let $\overline{\alpha} = \{\overline{A} : A \in \alpha\}$, then note that again $\mathbb{R}$-compact-open topology is obtained if α is replaced by $\overline{\alpha}$. Consequently, $RC_{\overline{\alpha}}(X) = RC_\alpha(X)$. From now on, α denotes a family of non-empty closed C-compact subsets of the set X.

The $\mathbb{R}$-compact-open topology on $C(X)$ denoted by $C_{rc}(X)$ introduced in [5] and the compact-open topology on $C(X)$ denoted by $C_k(X)$.

It is clear that $C_{rc}(X)$ is a subspace of $RC_{rc}(X)$. Also note that $C(X)$ is dense in $RC_{rc}(X)$[6].

Theorem 2.1. *If X is a submetrizable space,*
$C_k(X) = C_{rc}(X) = RC_{rc}(X)$.

Proof. $C_k(X) = C_{rc}(X)$ is given Theorem 4.14 in [5]. Since submetrizable space is a rc_f-space, $C(X) = RC(X)$ and consequently, $C_{rc}(X) = RC_{rc}(X)$. □

To metrizable and completely metrizable the $C_{rc}(X)$ space (similarly in the $RC_{rc}(X)$ space), the following results can be presented to be used in the next section.

A topological space is said to be hemi-C-compact if there exists a sequence of C-compact sets $\{A_n\}$ in X such that for any C-compact subset A of X, $A \subseteq A_n$ holds for some n. [5]

Theorem 2.2. *For a space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is metrizable..*
2. *$C_{rc}(X)$ is metrizable.*
3. *X is hemi-C-compact.*

Proof. Note $C_{rc}(X)$ is metrizable if and only if X is hemi-C-compact, see [5, Theorem 5.6]. Hence (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) This is immediate.

(3) $\Rightarrow$ (1) Here we need the well-known result which says that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable (see [18, page 119]). Now the locally convex topology on $RC(X)$ generated by the countable family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$ is metrizable and weaker than the $\mathbb{R}$-compact-open topology. But since for each C-compact set A in X, there exists A_n such that $A \subseteq A_n$, the locally convex topology generated by the family of seminorms $p_A : A \in QC(X)$, that is, the $\mathbb{R}$-compact-open topology is weaker than the topology generated by the family of seminorms $\{p_{A_n} : n \in \mathbb{N}\}$. Hence, $RC_{rc}(X)$ is metrizable. □

Theorem 2.3. *The space $C_{rc}(X)$ is uniformly complete if and only if X is a rc_f-space [6].*

Theorem 2.4. *The space $C_{rc}(X)$ is a completely metrizable if and only if X is a hemi-C-compact rc_f-space [6].*

Theorem 2.5. *For any space $RC_{rc}(X)$ is uniformly complete.*

Proof. Let (f_n) be a Cauchy net in $RC_{rc}(X)$. If A is a C-compact subset of X, then the net $\left(f_n\big|_A\right)$ is Cauchy in $RC_{rc}(A) = C_{rc}(A)$. But since $C_{rc}(A)$ is uniformly complete by Theorem 2.3, the net $\left(f_n\big|_A\right)$ converges to some f_A in $C_{rc}(A)$. Define $f : X \to \mathbb{R}$ by $f(x) = f_A(x)$ if $x \in A$. It can easily be seen that f is well defined and $f\big|_A = f_A$ for A for each C-compact subset A of X. Clearly $f \in RC(X)$. Also it is easy to see that (f_n) converges to f. □

Considering Theorem 2.2 and Theorem 2.3, it can give the following result.

Corollary 2.6. *For a space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is complete metrizable..*
2. *$RC_{rc}(X)$ is metrizable.*
3. *X is hemi-C-compact.*

3. Countability Properties

In this section, we study some countability properties such as $\aleph_0$-space, cosmic, separability, and second countability.

Recall that a family $\mathcal{N}$ of subsets of a topological space X is called a network if for every open set $U \subseteq X$ and point $x \in U$, there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq U$.

A space X is called cosmic if X is regular and has a countable network.

A family $\mathcal{N}$ of subsets of a topological space X is a k-network if for every open set $U \subseteq X$ and compact subset $K \subseteq X$, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ such that $K \subseteq \cup\mathcal{F} \subseteq U$.

A space X is $\aleph_0$-space if X is regular and has a countable k-network.

Note that metrizable separable space is $\aleph_0$-space and $\aleph_0$-space is cosmic space [11]. For more details, see [11] and [12].

It emerges from Proposition 10.2 in [11] that cosmic space is submetrizable. Now we have the following result.

Proposition 3.1. *For a cosmic space X, $RC(X) = C(X)$.*

Corollary 3.2. *$\aleph_0$-space X, $C_{rc}(X) = RC_{rc}(X)$.*

Proposition 3.3. *For $\aleph_0$-space X, $C_{rc}(X)$ and $RC_{rc}(X)$ is $\aleph_0$-space.*

Proof. If X is $\aleph_0$-space, being submetrizable, then $C_{rc}(X) = C_k(X)$ by Theorem 2.1, $C_{rc}(X) = RC_{rc}(X)$ by Proposition 3.1 and also $C_k(X)$ is $\aleph_0$-space by Lemma 2.3.6 in [11]. Consequently, $RC_{rc}(X)$ is $\aleph_0$-space. □

Corollary 3.4. *For separable metrizable space X, $RC_{rc}(X)$ is $\aleph_0$-space.*

Theorem 3.5. *For a space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is $\aleph_0$-space.*
2. *$RC_{rc}(X)$ is cosmic space.*
3. *$C_{rc}(X)$ is $\aleph_0$-space.*
4. *$C_{rc}(X)$ is cosmic space.*
5. *X is $\aleph_0$-space*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are well-known. By Proposition 3.3, (5) $\Rightarrow$ (1) and (5) $\Rightarrow$ (3) are clear. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are immediate, since every subspace of a cosmic space is cosmic.

(1) $\Rightarrow$ (5) Let $\mathcal{F}$ be a countable network for $RC_{rc}(X)$. For each $F \in \mathcal{F}$, let $F^* = \{x \in X : f(x) > 0\}$. Let us show that $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$ is a a countable k-network for X. Let $U \in \tau(X)$ and $A \in RC(X)$. Since X is Tychonoff, there exists a continuous function $f : X \to [0,1]$ such that $f(X \backslash U) = \{0\}$ and $f(A) = \{1\}$. Then $S(A, (0,\infty))$ is open in $C_{rc}(X)$ and $f \in S(A, (0,\infty))$, so there exists a $F \in \mathcal{F}$ such that $f \in F \subseteq S(A, (0,\infty))$. Therefore $A \subseteq F^*$. It remains to show that $F^* \subseteq U$. Let $x \in F^* \backslash U$. Since $x \notin U$, $f(x) = 0$. But this contradicts the fact that $x \in F^*$ and $f \in F$. Hence $A \subseteq F^* \subseteq U$ and so, $\mathcal{F}^*$ is a a countable k-network for X. Thus, X is an $\aleph_0$-space. Consequently, (4) $\Rightarrow$ (5). □

The space X is called σ-C-compact if there exists a sequence $\{A_n\}$ of C-compact sets in X such that $\bigcup_{n=1}^{\infty} A_n$ [5].

Theorem 3.6. *For σ-C-compact space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is separable.*
2. *$C_{rc}(X)$ is separable.*
3. *Every C-compact subset of X is metrizable.*
4. *X is a cosmic space.*
5. *X is submetrizable.*

Proof. (2) $\Rightarrow$ (1) If $C_{rc}(X)$ is separable, then X is submetrizable by Theorem 6.1 in [5] and so $C(X) = RC(X)$. It follows that $RC_{rc}(X)$ is separable.

(1) $\Rightarrow$ (3) Let A be C-compact subset of X. Then $C_q(A)$ is separable, since $C(X)$ is dense in $RC_{rc}(X)$ and so A is submetrizable. Since pseudocompactness is equal to C-compactness in submetrizable space and pseudocompact completely regular submetrizable space is metrizable [13, Corollary 2.7], then A is metrizable.

(3) $\Rightarrow$ (4) Since X is σ-C-compact, there exists a countable family $\{A_n : n \in \mathbb{N}\}$ of C-compact subsets of X such that $X = \bigcup_{n=1}^{\infty} A_n$. Each A_n, being compact and metrizable, is second countable and consequently, each A_n has a countable network $\mathcal{B}_n$. It is easy to show that $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a network for X, that is, X is a cosmic space.

(4) $\Rightarrow$ (5) Follows from Theorem 4.3.4 in [14].

(5) $\Rightarrow$ (2) First recall that since X is σ-C-compact, X is submetrizable if and only if X has a separable metrizable compression, that is, X has a weaker separable metrizable topology. Follows from Theorem 6.1 in [5]. □

Theorem 3.7. *For any space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is second countable.*
2. *$C_{rc}(X)$ is second countable.*
3. *X is hemi-C-compact and submetrizable.*
4. *X is hemi-C-compact and $\aleph_0$-space.*
5. *X is hemi-C-compact and cosmic space.*

Proof. (2) $\Rightarrow$ (1) If $C_{rc}(X)$ is second countable, then it is separable and submetrizable by Theorem 6.1 in [5]. Thus, $C(X) = RC(X)$. It follows that $RC_{rc}(X)$ is second countable.

(1) $\Rightarrow$ (3) If $RC_{rc}(X)$ is second countable, then it is metrizable as well as seprable. But then by Theorem 5.6 in [5], X is hemi-C-compact and consequently by Theorem 3.6, X is also submetrizable.

By Theorem 6.4 in [5], (3) $\Rightarrow$ (2) and the proof of (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is given Theorem 2.4.1 in [15]. □

Since locally compact $\aleph_0$-space is separable and metrizable [11], then the folloing result can be given.

Corollary 3.8. *For locally compact space X, the following statements are equivalent.*

1. *$RC_{rc}(X)$ is second countable.*
2. *$C_{rc}(X)$ is second countable.*
3. *X is Lindelöf and submetrizable.*
4. *X is $\aleph_0$-space.*
5. *X is cosmic space.*
6. *X is second countable.*

A space X is called Polish if it is homeomorphic to a separable complete metric space. As can be clearly seen, Polish space implies cosmic space.

Putting Theorem 2.4 and Theorem 3.7 together leads to the following theorem.

Theorem 3.9. *The space $C_{rc}(X)$ is Polish space if and only if X is a hemi-C-compact cosmic rc_f-space.*

Corollary 3.10. *The space $RC_{rc}(X)$ is Polish space if and only if X is a hemi-C-compact cosmic.*

Corollary 3.11. *For locally compact space X, the following statements are equivalent.*

1. *$C_{rc}(X)$ is Polish space.*
2. *$C_{rc}(X)$ is cosmic space.*
3. *X is a Polish space.*

4. Conclusion

In this study, we investigated the countability properties of the $\mathbb{R}$-compact-open topology on $RC(X)$, which is finer than the previously defined set-open topologies and will carry the studies on function spaces forward. Since comparing topologies will reveal the study's contribution to the literature, the obtained topologies are compared with compact-open topology and topology of uniform convergence. As it is known, countability plays a vital role in topological studies. Therefore, the countability properties of the obtained topologies will be analyzed in detail.

Topological properties not investigated in this work, or even R-compact-open topology on a more general set, can be defined and investigated for further work.

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