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Katugampola Fractional Integrals within the Class of Convex Functions

s-Konveks Fonksiyonlar Sınıfı için Katugampola Kesirli İntegraller

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Öz: Bu çalışmanın amacı; Katugampola kesirli integraller yardımıyla birinci mertebeden türevlerinin mutlak değeri s-konveks olan fonksiyonlar için Hermite-Hadamard tipli integral eşitsizlikler elde etmektir.

Anahtar Kelimeler — s-konveks fonksiyon, Hermite-Hadamard tipli eşitsizlikler, Katugampola kesirli integraller.

Abstract: The aim of this paper is to the Hermite-Hadamard type inequalities for functions whose first derivatives in absolute value is s-convex through the instrument of generalized Katugampola fractional integrals.

Keywords — s-convex function, Hermite-Hadamard type inequalities, Katugampola fractional integrals.

1. INTRODUCTION

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex function defined on the interval I of real number and $a, b \in I$, with $a < b$. Then the following double inequality is known in the literature as the Hermite-Hadamard's inequality for convex functions [7]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

The beginning fractional integral calculus accompanies the beginning of the integral calculus, developed by Riemann. It originates in the research of Liouville from 1832 related to practical technical problems. Now we point few stages in evolution of the fractional calculus, as needed in developing the new results. More details on the fractional differentiation and integration are in (see, [6, 11, 13]), for example. The Riemann-Liouville fractional integral is, from historic point of view, at the origin of the fractional calculus. It comes from the following Cauchy $n-$times iterative integration process,

$$\int_a^x dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_{n-1}} f(t_n)\,dt_n = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt,$$

for $n \in \mathbb{N}$.

By formally replacing n by a number $\alpha > 0$, one gets the classical Riemann-Liouville fractional integral, defined by:

Definition 1.1. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\left(J_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-t)^{\alpha-1} f(t)\,dt \qquad (x > a), \tag{1.2}$$

and

$$\left(J_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt \qquad (x < b), \tag{1.3}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$ is the Gamma function.

Hadamard developed in the second method of fractional integration based on the generalization of another iterative integral. Katugampola ([9] and [10]) considered the following iterative process in 2011:

$$\int_a^x t_1^{\rho} dt_1 \int_a^{t_1} t_2^{\rho} dt_2 \ldots \int_a^{t_{n-1}} t_n^{\rho} f(t_n)\,dt_n = \frac{(\rho+1)^{1-n}}{(n-1)!} \int_a^x \left(t^{\rho+1} - \tau^{\rho+1}\right)^{n-1} \tau^{\rho} f(\tau)\,d\tau,$$

for $n \in \mathbb{N}$. This generates Katugampola's concept of fractional integral, defined in [9] and also in [10].

Definition 1.2. *([9]) Let $f \in L[a,b]$, the left-sided Katugampola fractional integral ${}^{\rho}I_{a^+}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ is defined by*

$${}^{\rho}I_{a^+}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{\left(x^{\rho} - t^{\rho}\right)^{1-a}} f(t)\,dt, \quad x > a, \tag{1.4}$$

the right-sided Katugampola fractional integral ${}^{\rho}I_{b^-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ is defined by

$${}^{\rho}I_{b^-}^{\alpha} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{\left(t^{\rho} - x^{\rho}\right)^{1-a}} f(t)\,dt, \quad x < b. \tag{1.5}$$

Katugampola's operators are generalizations of A. Erdélyi and H. Kober operators introduced in 1940 (see [5] and [12]), as well. Other similar approaches on moving iterative integrals and derivatives into fractional framework in connection with theoretic and practical applications are in the mathematical literature of the last decade. For example, the results of Cristescu [4] in 2016.

Remark 1.1. *If $\rho = 1$ then the Katugampola fractional integrals become Riemann-Liouville fractional integrals.*

Now we reviewed some definitions and theorems which will be used in the proof of our main cumulative results.

Definition 1.3. *([1]) Let $s \in (0,1]$. A function $f:[0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\lambda x + (1-x)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$.

An $s-$convex function was introduced in Breckner's paper [1] and a number of properties and connections with $s-$convexity in the first sense were discussed in paper [8].

The main purpose of this paper is to introduce new type Hermite Hadamard and midpoint integral inequalities with the aid of generalized Katugampola fractional integral for $s-$convex functions and establish some results connected with the them (see for example, [2], [3] and [14]).

2. MAIN RESULTS

In this section, we will give Hermite-Hadamard type inequalities for the Katugampola fractional integrals by using $s-$convex functions.

Theorem 2.1. *Let $f:[a^\rho, b^\rho] \to \mathbb{R}$ be a function on with $0 \leq a < b$ and $f \in X_c^p(a^\rho, b^\rho)$. If f is also a $s-$convex on $[a,b]$, then the following inequalities hold:*

$$\frac{2^s}{\rho} f\left(\frac{a^\rho + b^\rho}{2}\right) \tag{2.1}$$

$$\leq \frac{2^\alpha \rho^{\alpha-1}\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+} f(b^\rho) + {}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-} f(a^\rho)\right]$$

$$\leq 2^{-s}\left[f(a^\rho) + f(b^\rho)\right]\left[\frac{1}{\rho(\alpha+s)} + \frac{2^{(\alpha+s)} B_{\frac{1}{2}}(\alpha, s+1)}{\rho}\right];$$

$$\left[\left(Re\left(2^{\frac{1}{\rho}} \geq 1\right) \vee Re\left(2^{\frac{1}{\rho}} \leq 0\right) \vee 2^{\frac{1}{\rho}} \notin \mathrm{R}\right) \wedge Re(\rho) > 0 \wedge Re(\alpha\rho) > 0\right]$$

where

$$B_{\frac{1}{2}}(\alpha, s+1) = \int_0^{\frac{1}{2}} u^{\alpha+1}(1-u)^s du$$

the fractional integrals are considered for the function $f\left(x^{\rho}\right)$ and evaluated at a and b, respectively.

Proof. Since f is $s-$convex function on $[a,b]$, we have for $x, y \in [a,b]$

$$f\left(\frac{x^{\rho}+y^{\rho}}{2}\right) \leq \frac{f\left(x^{\rho}\right)+f\left(y^{\rho}\right)}{2^{s}}$$

for $x^{\rho}=\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}$ and $y^{\rho}=\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}$, we obtain

$$2^{s} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \leq f\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)+f\left(\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}\right) \tag{2.2}$$

Multiplying both sides of (2.2) by $t^{\alpha\rho-1}$, $\alpha>0$ and then integrating with respect to t over $[0,1]$, we get

$$\frac{2^{s}}{\rho} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \tag{2.3}$$

$$\leq \int_0^1 t^{\alpha\rho-1} f\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)dt+\int_0^1 t^{\alpha\rho-1} f\left(\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}\right)dt$$

$$=2^{\alpha}\left[\int_0^{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{\frac{1}{\rho}}}\left(\frac{x^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\frac{x^{\rho-1}}{\left(b^{\rho}-a^{\rho}\right)} f\left(x^{\rho}\right)dx\right.$$

$$\left.+\int_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{\frac{1}{\rho}}}^{b}\left(\frac{b^{\rho}-x^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\frac{x^{\rho-1}}{\left(b^{\rho}-a^{\rho}\right)} f\left(x^{\rho}\right)dx\right]$$

$$=\frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{\left(b^{\rho}-a^{\rho}\right)^{\alpha}}\left[{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{+}} f\left(b^{\rho}\right)+{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{-}} f\left(a^{\rho}\right)\right]$$

and the first inequality is proved. For the proof of the second inequality (2.3), we first note that if f is a $s-$convex function, it yields

$$f\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right) \leq \frac{t^{\rho s}}{2^{s}} f\left(a^{\rho}\right)+\frac{\left(2-t^{\rho}\right)^{s}}{2^{s}} f\left(b^{\rho}\right)$$

and

$$f\left(\frac{2-t^{\rho}}{2}a^{\rho}+\frac{t^{\rho}}{2}b^{\rho}\right)\leq\frac{\left(2-t^{\rho}\right)^{s}}{2^{s}}f\left(a^{\rho}\right)+\frac{t^{\rho s}}{2^{s}}f\left(b^{\rho}\right)$$

By adding these inequalities together, one has the following inequality:

$$f\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)+f\left(\frac{2-t^{\rho}}{2}a^{\rho}+\frac{t^{\rho}}{2}b^{\rho}\right)\leq 2^{-s}\left[t^{\rho s}+\left(2-t^{\rho}\right)^{s}\right]\left[f\left(a^{\rho}\right)+f\left(b^{\rho}\right)\right] \tag{2.4}$$

Then multiplying both sides of and (2.4) by $t^{\alpha\rho-1}$ and integrating the resulting inequality with respect to t over $[0,1]$ we obtain

$$\frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{\left(b^{\rho}-a^{\rho}\right)^{\alpha}}\left[{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{+}}f\left(b^{\rho}\right)+{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{-}}f\left(a^{\rho}\right)\right]$$

$$\leq 2^{-s}\left[f\left(a^{\rho}\right)+f\left(b^{\rho}\right)\right]\left[\frac{1}{\rho(\alpha+s)}+\frac{2^{(\alpha+s)}B_{\frac{1}{2}}(\alpha,s+1)}{\rho}\right],$$

$$\left[\left(Re\left(2^{\frac{1}{\rho}}\geq 1\right)\vee Re\left(2^{\frac{1}{\rho}}\leq 0\right)\vee 2^{\frac{1}{\rho}}\notin \mathbb{R}\right)\wedge Re(\rho)>0\wedge Re(\alpha\rho)>0\right]$$

In this way the proof is completed.

Corollary 2.1. *If we write $\rho=1$ in inequality (2.1), we obtain;*

$$2^{s}f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]$$

$$\leq 2^{-s}\left[f(a)+f(b)\right]\left[\frac{1}{(\alpha+s)}+2^{(\alpha+s)}B_{\frac{1}{2}}(\alpha,s+1)\right]$$

with $Re(\alpha)>0$.

Remark 2.1. *Choosing $s=1$ in Corollary 2.1, we obtain following inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

which was given by Sarıkaya and Yıldırım in [15].

Now, we need to give a lemma for differentiable functions which help us to prove our main theorems.

Lemma 2.1. *Let* $f:[a^\rho,b^\rho]\to\mathbb{R}$ *be a differentiable mapping on* (a^ρ,b^ρ) *with* $0\le a<b$, *then the following equality holds:*

$$\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+} f(b^\rho)+{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-} f(a^\rho)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right) \tag{2.5}$$

$$=\frac{(b^\rho-a^\rho)\rho}{4}\left[\int_0^1 t^{\alpha\rho} f'\left(\frac{t^\rho}{2}a^\rho+\frac{2-t^\rho}{2}b^\rho\right)dt-\int_0^1 t^{\alpha\rho} f'\left(\frac{t^\rho}{2}b^\rho+\frac{2-t^\rho}{2}a^\rho\right)dt\right]$$

Proof. Integrating by parts gives

$$H_1=\int_0^1 t^{\alpha\rho} f'\left(\frac{t^\rho}{2}a^\rho+\frac{2-t^\rho}{2}b^\rho\right)dt \tag{2.6}$$

$$=\frac{-2}{\rho(b^\rho-a^\rho)}f\left(\frac{a^\rho+b^\rho}{2}\right)+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{\rho^{1-\alpha}(b^\rho-a^\rho)^{\alpha+1}}\,{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+} f(b^\rho),$$

and

$$H_2=\int_0^1 t^{\alpha\rho} f'\left(\frac{t^\rho}{2}b^\rho+\frac{2-t^\rho}{2}a^\rho\right)dt \tag{2.7}$$

$$=\frac{2}{\rho(b^\rho-a^\rho)}f\left(\frac{a^\rho+b^\rho}{2}\right)-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{\rho^{1-\alpha}(b^\rho-a^\rho)^{\alpha+1}}\,{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-} f(a)$$

by subtracting equation (2.7) from (2.6), we have

$$H_1-H_2=\frac{-4}{\rho(b^\rho-a^\rho)}f\left(\frac{a^\rho+b^\rho}{2}\right)$$

$$+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{\rho^{1-\alpha}(b^\rho-a^\rho)^{\alpha+1}}\left[{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+} f(b)+{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-} f(a)\right].$$

By re-arranging the last equality above, we get the desired result.

Theorem 2.2. *Let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable mapping on (a^ρ,b^ρ) with $0\le a<b$. If $|f'|$ is $s-$convex on $[a^\rho,b^\rho]$, then the following inequality holds:*

$$\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+}f(b^\rho)+{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-}f(a^\rho)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right| \tag{2.8}$$

$$\le\frac{(b^\rho-a^\rho)}{2^{s+2}}\left[|f'(a^\rho)|+|f'(b^\rho)|\right]\left[\frac{\rho}{\rho(\alpha+s)+1}+2^{\left(\alpha+s+\frac{1}{\rho}\right)}B_{\frac{1}{2}}\left(\alpha+\frac{1}{\rho},s+1\right)\right];$$

$$[Re(\rho)\ge 0\wedge Re(\alpha\rho)>-1].$$

$B_{\frac{1}{2}}$ is defined as in Theorem 2.1.

Proof. Taking modulus of (2.5) and using $s-$convexity of $|f'|$, we have

$$\left|\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^+}f(b^\rho)+{}^\rho I^\alpha_{\left(\frac{a^\rho+b^\rho}{2}\right)^-}f(a^\rho)\right]-f\left(\frac{a^\rho+b^\rho}{2}\right)\right|$$

$$\le\frac{(b^\rho-a^\rho)\rho}{4}\left[\int_0^1 t^{\alpha\rho}\left|f'\left(\frac{t^\rho}{2}a^\rho+\frac{2-t^\rho}{2}b^\rho\right)\right|dt\right.$$

$$\left.+\int_0^1 t^{\alpha\rho}\left|f'\left(\frac{t^\rho}{2}b^\rho+\frac{2-t^\rho}{2}a^\rho\right)\right|dt\right]$$

$$\le\frac{(b^\rho-a^\rho)\rho}{4}\left[\int_0^1 t^{\alpha\rho}\left[\left(\frac{t^\rho}{2}\right)^s|f'(a^\rho)|+\left(\frac{2-t^\rho}{2}\right)^s|f'(b^\rho)|\right]dt\right.$$

$$\left.\int_0^1 t^{\alpha\rho}\left[\left(\frac{t^\rho}{2}\right)^s|f'(b^\rho)|+\left(\frac{2-t^\rho}{2}\right)^s|f'(a^\rho)|\right]dt\right]$$

$$\le\frac{(b^\rho-a^\rho)}{2^{s+2}}\left[|f'(a^\rho)|+|f'(b^\rho)|\right]\left[\frac{\rho}{\rho(\alpha+s)+1}+2^{\left(\alpha+s+\frac{1}{\rho}\right)}B_{\frac{1}{2}}\left(\alpha+\frac{1}{\rho},s+1\right)\right],$$

$$[Re(\rho)\ge 0\wedge Re(\alpha\rho)>-1].$$

where $B_{\frac{1}{2}}$ is defined above. Thus, the proof is completed.

Corollary 2.2. *If we write* $\rho=1$ *in inequality (2.8), we obtain;*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{2^{s+2}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left[\frac{1}{\alpha+s+1}+2^{(\alpha+s+1)}B_{\frac{1}{2}}(\alpha+1,s+1)\right].$$

Remark 2.2. *Choosing* $s=1$ *in Corollary 2.2, we obtain following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{4(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

which was given by Sarıkaya and Yıldırım in [15].

Theorem 2.3. *Let* $f:[a^{\rho},b^{\rho}]\to\mathbb{R}$ *be a differentiable mapping on* (a^{ρ},b^{ρ}) *with* $0\leq a<b$. *If* $\left|f'\right|^{q}$ *,* $q>1$*, is* $s-$*convex on* $[a^{\rho},b^{\rho}]$*, then the following inequality holds:*

$$\left|\frac{2^{\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{+}}f(b^{\rho})+{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{-}}f(a^{\rho})\right]-f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right| \tag{2.9}$$

$$\leq\frac{(b^{\rho}-a^{\rho})\rho}{4}\left(\frac{1}{\alpha\rho p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f'(a^{\rho})\right|^{q}}{2^{s}(\rho s+1)}+\frac{2^{\left(\frac{1}{\rho}+s\right)}B_{\frac{1}{2}}\left(\frac{1}{\rho},s+1\right)\left|f'(b^{\rho})\right|^{q}}{2^{s}\rho}\right]^{\frac{1}{q}}$$

$$\times\left[\frac{2^{\left(\frac{1}{\rho}+s\right)}B_{\frac{1}{2}}\left(\frac{1}{\rho},s+1\right)\left|f'(a^{\rho})\right|^{q}}{2^{s}\rho}+\frac{\left|f'(b^{\rho})\right|^{q}}{2^{s}(\rho s+1)}\right]^{\frac{1}{q}},\quad (Re(\rho)\geq 0)$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $B_{\frac{1}{2}}$ is defined as in Theorem 2.1.

Proof. Taking modulus of (2.5) and using well-known Hölder inequality, we obtain

$$\left|\frac{2^{\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{\left(b^{\rho}-a^{\rho}\right)^{\alpha}}\left[{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{+}}f\left(b^{\rho}\right)+{}^{\rho}I^{\alpha}_{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{-}}f\left(a^{\rho}\right)\right]-f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right| \tag{2.10}$$

$$\leq \frac{\left(b^{\rho}-a^{\rho}\right)\rho}{4}\left[\int_0^1 t^{\alpha\rho}\left|f'\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)\right|dt\right.$$

$$\left.+\int_0^1 t^{\alpha\rho}\left|f'\left(\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}\right)\right|dt\right]$$

$$\leq \frac{\left(b^{\rho}-a^{\rho}\right)\rho}{4}\left(\int_0^1 t^{\alpha\rho p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left|f'\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

Since $\left|f'\right|^{q}$, $q>1$, is $s-$convex, we have

$$\int_0^1\left|f'\left(\frac{t^{\rho}}{2}a^{\rho}+\frac{2-t^{\rho}}{2}b^{\rho}\right)\right|^{q}dt \tag{2.11}$$

$$\leq \int_0^1\left[\left(\frac{t^{\rho}}{2}\right)^{s}\left|f'\left(a^{\rho}\right)\right|^{q}+\left(\frac{2-t^{\rho}}{2}\right)^{s}\left|f'\left(b^{\rho}\right)\right|^{q}\right]dt$$

$$=\frac{\left|f'\left(a^{\rho}\right)\right|^{q}}{2^{s}\left(\rho s+1\right)}+\frac{2^{\left(\frac{1}{\rho}+s\right)}B_{\frac{1}{2}}\left(\frac{1}{\rho},s+1\right)\left|f'\left(b^{\rho}\right)\right|^{q}}{2^{s}\rho},\ \left(Re(\rho)\geq 0\right)$$

and similarly

$$\int_0^1\left|f'\left(\frac{t^{\rho}}{2}b^{\rho}+\frac{2-t^{\rho}}{2}a^{\rho}\right)\right|^{q}dt \tag{2.12}$$

$$\leq \frac{2^{\left(\frac{1}{\rho}+s\right)}B_{\frac{1}{2}}\left(\frac{1}{\rho},s+1\right)\left|f'\left(a^{\rho}\right)\right|^{q}}{2^{s}\rho}+\frac{\left|f'\left(b^{\rho}\right)\right|^{q}}{2^{s}\left(\rho s+1\right)},\ \left(Re(\rho)\geq 0\right)$$

By substituting inequalities (2.11) and (2.12) into (2.10), we get the desired result (2.9).

Corollary 2.3. *If we write* $\rho=1$ *in inequality (2.9), we obtain;*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^{q}}{2^{s}(s+1)}+\frac{2^{(s+1)}B_{\frac{1}{2}}(1,s+1)|f'(b)|^{q}}{2^{s}}\right]^{\frac{1}{q}}$$

$$\times\left[\frac{2^{(s+1)}B_{\frac{1}{2}}(1,s+1)|f'(a)|^{q}}{2^{s}}+\frac{|f'(b)|^{q}}{2^{s}(s+1)}\right]^{\frac{1}{q}}.$$

Remark 2.3. *Choosing* $s=1$ *in Corollary 2.3, we obtain following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^{q}}{4}+\frac{3|f'(b)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^{q}}{4}+\frac{3|f'(a)|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(b-a)}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right]$$

which is the same result given by Sarıkaya and Yıldırım in [15].

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