TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 1, 45–53
ISSN: 2587–0971

https://www.tjoscience.com

Coefficient estimates and Fekete-Sezegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function of complex order

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Abstract. The main focus of this paper is to introduce and examine a new subclass of the analytic and univalent functions defined on the open unit disk in the complex plane, which are subordinated to sine hyperbolic function with complex order. Here, we obtain some coefficient estimates for the initial two coefficients and examine the Fekete-Szegö problem for the defined class of analytic and univalent functions.

1. Introduction and preliminaries

In this section, we give some basic information which we will use in the proof of the main results and to discuss the studies known in the literature related to our subject

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{U})$ denote the class of all analytic functions in $\mathbb{U}$. By A, we will denote the class of the functions $f \in H(\mathbb{U})$ given by the following series expansion, which satisfied the conditions $f(0) = 0$ and $f'(0) - 1 = 0$

Let A denote the class of all complex valued functions f given by

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + ... + = z + \sum_{n=2}^{\infty} a_nz^n, a_n \in \mathbb{C}. \tag{1}$$

As is known that the subclass of univalent functions of A is denoted by S in the literature. This class was introduced by Köebe [10] first time and has become the core ingredient of advanced research in this field. Bieberbach [4] published a paper in which the famous coefficient hypothesis was proposed. This conjecture states that if $f \in S$ and has the series form (1), then $|a_n| \leq n$ for all $n \geq 2$. Many researchers worked hard to solve this problem. But for the first time this long-lasting conjecture solved by de-Branges [5] in 1985.

It is well-know that a univalent function $f \in S$ is called a starlike and convex function, if this function maps open unit disk $\mathbb{U}$ onto the star shaped and convex shaped domain of the complex plane, respectively. The set of all starlike and convex functions in $\mathbb{U}$, which satisfies the following conditions are denoted by S^* and C, respectively

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ and } \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in \mathbb{U}\,;$$

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Received: 12 February 2024; Accepted: 2 April 2024; Published: 30 April 2024

Keywords. Starlike function, sine hyperbolic function, coefficient estimate, Fekete-Szegö problem, complex order

2010 *Mathematics Subject Classification*. 30C45, 30C50, 30C55, 30C80

Cited this article as: Mustafa, N., & Nezir, V. (2024). Coefficient estimates and Fekete-Sezegö problem for certain subclass of analytic and univalent functions associated with sine hyperbolic function of complex order, Turkish Journal of Science, 9(1), 45-53.

$$S^{*}=\left\{f\in\ S:\Re\left(\frac{zf'(z)}{f(z)}\right)>0,z\in\mathbb{U}\right\},$$

and

$$C=\left\{f\in S:\Re\left(\frac{(zf'(z))'}{f'(z)}\right)>0,z\in\mathbb{U}\right\}.$$

Some of the important and well-investigated subclass of S include the classes $S^{*}(\alpha)$ and $C(\alpha)$ given below, which are called of the starlike and convex functions of order α ($\alpha\in[0,1)$) respectively

$$S^{*}(\alpha)=\left\{f\in\ S:\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,z\in\mathbb{U}\right\},$$

$$C(\alpha)=\left\{f\in S:\Re\left(\frac{(zf'(z))'}{f'(z)}\right)>\alpha,z\in\mathbb{U}\right\}$$

It is well-known that an analytical function ω satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$ is called Schwartz function. Let's $f,g\in H(\mathbb{U})$, then it is said that f is subordinate to g and denoted by $f\prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.

For $\beta\in[0,1]$, in 1992, Ma and Minda [12] using subordination terminology presented unified version of the classes $S^{*}(\varphi)$ and $C(\varphi)$ as follows

$$(S^{*}\vee C)(\varphi)=\left\{f\in S:(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}\prec\varphi(z),z\in\mathbb{U}\right\},$$

where $\varphi(z)$ is a univalent function with $\varphi(0)$, $\varphi'(0)>0$ and the region $\varphi(\mathbb{U})$ is star-shaped about the point and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z)=1+b_1z+b_2z^2+b_3z^2+...=1+\sum_{n=1}^{\infty}b_nz^n,b_1>0.$$

In the past few years, numerous subclasses of S have been introduced as special choices of the function φ (see for example [5-23]).

Finding bounds for the function coefficients in a given collection is one of the most fundamental problems in geometric function theory.

The first order of Hankel determinant of the function $f\in S$ defined by

$$H_{2,1}=\begin{vmatrix}1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2$$

and is known as the Fekete-Szegö functional. For $\mu\in\mathbb{C}$ or $\mu\in\mathbb{R}$, the functional $H_{2,1}(f,\mu)=a_3-\mu a_2^2$ is known as the generalized Fekete-Szegö functional [8]. Estimating the upper bound of $\left|a_3-\mu a_2^2\right|$ is known as the Fekete-Szegö probblem in the theory of analytic functions.

Now by using the definition of subordination, we indroduce a new subclass of analytic and univalent functions associated with sine hyporbilic function with comlpex order.

Definition 1.1. *For $\tau\in\mathbb{C}-\{0\}$ and $\beta\in[0,1]$ a function $f\in S$ is said to be in the class $\left(S^{*}_{\sinh}\vee C_{\sinh}\right)(\tau,\beta)$, if the folloing condition is satisfied*

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\}\prec 1+\sinh z,z\in\mathbb{U}$$

that is

$$\left(S^{*}_{\sinh}\vee C_{\sinh}\right)(\tau,\beta)=\left\{\begin{array}{c} f\in S:(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\}\\ \prec 1+\sinh z,z\in\mathbb{U}\end{array}\right\}$$

Remark 1.2. . *Taking $\beta = 0$ and $\beta = 1$ in the Definition 1.1, we have class*

$$\left(S^*_{\sinh}\right)(\tau) = \left\{ f \in S : \left\{ 1 + \frac{1}{\tau}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} \prec 1 + \sinh z, z \in \mathbb{U} \right\}$$

and

$$\left(C_{\sinh}\right)(\tau) = \left\{ f \in S : \left\{ 1 + \frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)} - 1\right]\right\} \prec 1 + \sinh z, z \in \mathbb{U} \right\},$$

respectively.

Remark 1.3. *Taking $\tau = 1$ in the Definition 1.1, we have class*

$$\left(S^*_{\sinh} \vee C_{\sinh}\right)(\beta) = \left\{ f \in S : (1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{(zf'(z))'}{f'(z)} \prec 1 + \sinh z, z \in \mathbb{U} \right\}.$$

Remark 1.4. *Taking $\tau = 1$, $\beta = 0$ and $\beta = 1$ in the Definition 1.1 or $\tau = 1$ in Remark 1.2, we have classes*

$$S^*_{\sinh} = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec 1 + \sinh z, z \in \mathbb{U} \right\}$$

and

$$C_{\sinh} = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \prec 1 + \sinh z, z \in \mathbb{U} \right\}$$

respectively.

Remark 1.5. *It is useful to remind that the classes $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\beta)$, $S^*_{\sinh}$,$C_{\sinh}$, $\left(S^*_{\sinh}\right)(\tau)$ and $\left(C_{\sinh}\right)(\tau)$ are reviewed in [15], [16], [18], [17] and [19], respectively.*

Let P be the class of analytic functions $\mathbb{U}$ in satisfied the conditions $p(0) = 1$ and $Re(p(z) > 0)$, $z \in \mathbb{U}$, which from the subordination principle easily can written

$$P = \left\{ p \in A : p(z) \prec \frac{1+z}{1-z}, z \in \mathbb{U} \right\}$$

where $p(z)$ has the series expansion of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^2 + ... = 1 + \sum_{n=1}^{\infty} p_n z^n, z \in \mathbb{U}.$$

The class P defined above is knowwn as the class Caratheodory functions [14].

Now, let us present some necessary lemmas known in the literature for the proof of our main results.

Lemma 1.6. *([8]). Let the function $p(z)$ belong in the class P. Then,*
$\left|p_n\right| \le 2$ for each $n \in \mathbb{N}$ and $\left|p_n - \lambda p_k p_{n-k}\right| \le 2$ for $n, k \in \mathbb{N}$, $n > k$ and $\lambda \in [0,1]$.
The equalities holds for

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 1.7. *([8]) Let the an analytic function $p(z)$ be of the form (1), then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)y$$

for $x, y \in \mathbb{C}$ with $|x| \le 1$ and $\left|y\right| \le 1$.

2. Coefficient estimates for the class $\left(S^{*}_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$

In this section, we give upper bound estimates for the initial two coefficients of the functions belonging to the class $\left(S^{*}_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$. The following theorem is related to this.

Theorem 2.1. *Let the function $f \in A$ given by (1) belong to the class $\left(S^{*}_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$. Then*

$$|a_2| \leq \frac{|\tau|}{1+\beta} \text{ and } |a_3| \leq \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & if \quad |\tau| \leq \frac{(1+\beta)^2}{1+3\beta}, \\ \frac{(1+3\beta)|\tau|}{(1+\beta)^2} & if \quad |\tau| \geq \frac{(1+\beta)^2}{1+3\beta}. \end{cases}$$

Proof. Let $f \in \left(S^{*}_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$, $\tau \in \mathbb{C} - \{0\}$ and $\beta \in [0,1]$, then there exsists a Schwartz function ω, such that

$$(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\} \prec 1+\sinh\omega(z).$$

By writing the Caratheodory function in terms of the Schwartz function ω, we have

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + ...$$

It follows from that

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + ... \tag{2}$$

From the series expansion (1) of the function $f(z)$, we can write

$$\begin{aligned} &(1-\beta)\left\{1+\frac{1}{\tau}\left[\frac{zf'(z)}{f(z)}-1\right]\right\}+\beta\left\{1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right]\right\} \\ = \;& 1+\frac{1}{\tau}\left\{(1+\beta)a_2 z + \left[2(1+2\beta)a_3 - (1+3\beta)a_2^2\right]z^2 + ...\right\}. \end{aligned} \tag{3}$$

Since

$$\sinh z = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ... \tag{4}$$

using the series expansion (2) of the function $\omega(z)$, we have

$$1+\sinh\omega(z) = 1 + \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + ... \tag{5}$$

By equalizing (3) and (5), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for two initial coefficients of the function f

$$a_2 = \frac{\tau p_1}{2(1+\beta)}, \tag{6}$$

$$a_3 = \frac{\tau}{8(1+2\beta)}\left\{2p_2 + \left[\frac{(1+3\beta)\tau}{(1+\beta)^2} - 1\right]p_1^2\right\}. \tag{7}$$

Using Lemma 1.6, from the equalities (6) we can easily see that

$$|a_2| \leq \frac{|\tau|}{1+\beta}.$$

Using Lemma 1.7, the equality (7) can written as follows

$$a_3 = \frac{\tau}{8(1+2\beta)}\left[\frac{(1+3\beta)\tau}{(1+\beta)^2}p_1^2 + \left(4-p_1^2\right)x\right].$$

where $x \in \mathbb{C}$ with $|x| \leq 1$. Applying triangle inequality, from the last equality we obtain

$$|a_3| \leq \frac{|\tau|}{8(1+2\beta)}\left[\frac{(1+3\beta)|\tau|}{(1+\beta)^2}t^2 + \left(4-t^2\right)\xi\right].$$

where $\xi = |x|$ and $t = \left|p_1\right|$. If we maximize the function

$$\varphi(\xi) = \frac{(1+3\beta)|\tau|}{(1+\beta)^2}t^2 + \left(4-t^2\right)\xi, \xi \in [0,1],$$

we write

$$|a_3| \leq \frac{|\tau|}{8(1+2\beta)}\left[\frac{(1+3\beta)|\tau|}{(1+\beta)^2}t^2 + 4\right], t \in [0,2].$$

From this, immediately obtained the desired estimate for $|a_3|$.
Thus, the proof of theorem is completed.
Taking $\tau = 1$ in Theorem 2.1, we obtain the following results for $|a_2|$ and $|a_3|$ obtained in [15]. □

Theorem 2.2. *Let the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\beta)$. Then*

$$|a_2| \leq \frac{1}{1+\beta} \text{ and } |a_3| \leq \frac{1+3\beta}{2(1+2\beta)(1+\beta)^2}.$$

Setting $\beta = 0$ and $\beta = 1$ in Theorem 2.1, we obtain the following results obtained in [17] and [19], respectiely.

Corollary 2.3. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}(\tau)$, then*

$$|a_2| \leq |\tau| \text{ and } |a_3| \leq \frac{|\tau|}{2}\begin{cases} 1 & if \quad |\tau| \leq 1, \\ |\tau| & if \quad |\tau| \geq 1.. \end{cases}.$$

Corollary 2.4. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}(\tau)$, then*

$$|a_2| \leq \frac{|\tau|}{2} \text{ and } |a_3| \leq \frac{|\tau|}{6}\begin{cases} 1 & if \quad |\tau| \leq 1, \\ |\tau| & if \quad |\tau| \geq 1. \end{cases}$$

Setting $\beta = 0$ and $\beta = 1$ in Theorem2.2 or in Corollary 2.3 and Corollary 2.4, we obtain the following results obtained in [16] and [18], respctively.

Corollary 2.5. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}$, then*

$$|a_2| \leq 1 \text{ and } |a_3| \leq \frac{1}{2}.$$

Corollary 2.6. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}$, then*

$$|a_2| \leq \frac{1}{2} \text{ and } |a_3| \leq \frac{1}{6}.$$

3. The Fekete-Szegö problem for the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau, \beta)$

In this section, we examine the Fekete-Szegö problem for the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau, \beta)$. In the following theorem, we give a inequality for the Fekete-Szegö functionel of the function belonging to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau, \beta)$.

Theorem 3.1. *Let the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau, \beta)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & if \quad \left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| \leq \frac{(1+\beta)^2}{2|\tau|(1+2\beta)}, \\ \frac{1+3\beta-2(1+2\beta)\mu|\tau|}{(1+\beta)^2} & if \quad \left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| \geq \frac{(1+\beta)^2}{2|\tau|(1+2\beta)}. \end{cases}$$

Proof. *Let $f \in \left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau, \beta)$, $\tau \in \mathbb{C} - \{0\}$, $\beta \in [0,1]$ and $\mu \in \mathbb{C}$. Then from the equalities (6) and (7), we can write*

$$a_3 - \mu a_2^2 = \frac{\tau}{4}\left\{\frac{p_2}{1+2\beta} + \left\{\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] - \frac{1}{2(1+2\beta)}\right\} p_1^2\right\}.$$

Using Lemma 1.7, last equality we can write as follows

$$a_3 - \mu a_2^2 = \frac{\tau}{4}\left[\frac{\tau}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] p_1^2 + \frac{4-p_1^2}{2(1+2\beta)} x\right], \tag{8}$$

for $x \in \mathbb{C}$ with $|x| \leq 1$. From here, applying triangle inequality we have

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{4}\left[\frac{|\tau|}{(1+\beta)^2}\left[\frac{1+3\beta}{2(1+2\beta)} - \mu\right] t^2 + \frac{4-t^2}{2(1+2\beta)}\xi\right],$$

with $t = \left|p_1\right| \in [0,2]$ and $\xi = |x|$. By maximizing the function

$$\psi(\xi) = \frac{|\tau|}{(1+\beta)^2}\left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| t^2 + \frac{4-t^2}{2(1+2\beta)}\xi, \xi \in [0,1],$$

we obtain the following inequality

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{4}\left\{\left[\frac{|\tau|}{(1+\beta)^2}\left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| - \frac{1}{2(1+2\beta)}\right] t^2 + \frac{2}{1+2\beta}\right\}, t \in [0,2]. \tag{9}$$

The result of the theorem is easily obtained from the inequality (9). Thus, the proof of theorem is completed. □

Taking $\tau = 1$ in Theorem 3.1, we obtain the following result for the Fekete-Szegö inequality obtained in [15].

Theorem 3.2. *Let the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\beta)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2(1+2\beta)} \begin{cases} 1 & if \quad \left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| \leq \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \frac{|1+3\beta-2(1+2\beta)\mu|}{(1+\beta)^2} & if \quad \left|\frac{1+3\beta}{2(1+2\beta)} - \mu\right| \geq \frac{(1+\beta)^2}{2(1+2\beta)}. \end{cases}$$

Setting $\beta = 0$ and $\beta = 1$ in Theorem 3.1, we obtain the following results obtained in [17] and [19], respectively.

Corollary 3.3. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}(\tau)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & if \quad \left|1-2\mu\right||\tau| \leq 1, \\ \left|1-2\mu\right||\tau| & if \quad \left|1-2\mu\right||\tau| \geq 1. \end{cases}$$

Corollary 3.4. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}(\tau)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\tau|}{6} \begin{cases} 1 & if \quad \left|2-3\mu\right||\tau| \leq 2, \\ \frac{\left|2-3\mu\right||\tau|}{2} & if \quad \left|2-3\mu\right||\tau| \geq 2. \end{cases}$$

Setting $\beta = 0$ and $\beta = 1$ in Theorem 3.2 or $\tau = 1$ in Corollary 3.3 and Corollary 3.4, we obtain the following results obtained in [16] and [18], respctively.

Corollary 3.5. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2} \begin{cases} 1 & if \quad \left|1-2\mu\right| \leq 1, \\ \left|1-2\mu\right| & if \quad \left|1-2\mu\right| \geq 1. \end{cases}$$

Corollary 3.6. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}(\tau)$ and $\mu \in \mathbb{C}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{6} \begin{cases} 1 & if \quad \left|2-3\mu\right| \leq 2, \\ \frac{\left|2-3\mu\right|}{2} & if \quad \left|2-3\mu\right| \geq 2. \end{cases}$$

In the case $\mu \in \mathbb{R}$, Theorem 3.1 is given as below.

Theorem 3.7. *Let the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$ and $\mu \in \mathbb{R}$. Then,*

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| \\ \leq \quad & \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & if \quad \frac{(1+3\beta)|\tau|-(1+\beta)^2}{2|\tau|(1+2\beta)} \leq \mu \leq \frac{(1+3\beta)|\tau|-(1+\beta)^2}{2|\tau|(1+2\beta)}, \\ \frac{1+3\beta-2(1+2\beta)\mu|\tau|}{(1+\beta)^2} & if \quad \begin{array}{c} \mu \leq \frac{(1+3\beta)|\tau|-(1+\beta)^2}{2|\tau|(1+2\beta)} \\ or \ \frac{(1+3\beta)|\tau|-(1+\beta)^2}{2|\tau|(1+2\beta)} \leq \mu. \end{array} \end{cases} \end{aligned}$$

The proof of Theorem 3.7 is done similarly to the proof of Theorem 2.1.

Setting $\tau = 1$ in Theorem 3.7, we obtain the following result for the Fekete-Szegö inequality obtained in [15].

Theorem 3.8. *Let the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\beta)$ and $\mu \in \mathbb{R}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2(1+2\beta)} \begin{cases} 1 & if \quad \frac{(1-\beta)\beta}{2(1+2\beta)} \leq \mu \leq \frac{\beta^2+5\beta+2}{2(1+2\beta)}, \\ \frac{1+3\beta-2(1+2\beta)\mu}{(1+\beta)^2} & if \quad \mu \leq \frac{(1-\beta)\beta}{2(1+2\beta)} \ or \ \frac{\beta^2+5\beta+2}{2(1+2\beta)} \leq \mu. \end{cases}$$

Taking $\mu = 0$ and $\mu = 1$ in Theorem 3.7, we get the second result of Theorem 2.1 and the following result, respectively.

Corollary 3.9. *If the function $f \in A$ given by (1) belong to the class $\left(S^*_{\sinh} \vee C_{\sinh}\right)(\tau,\beta)$. Then,*

$$\left|a_3 - a_2^2\right| \leq \frac{|\tau|}{2(1+2\beta)} \begin{cases} 1 & if \quad |\tau| \leq 1+\beta, \\ \frac{|\tau|}{1+\beta} & if \quad |\tau| \geq 1+\beta. \end{cases}$$

Taking $\beta = 0$ and $\beta = 1$ in the Corollary 3.9, we get the following resualts obtained in [17] and [19], repectively.

Corollary 3.10. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}(\tau, \beta)$. Then,*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{2} \begin{cases} 1 & if \quad |\tau| \le 1, \\ |\tau| & if \quad |\tau| \ge 1. \end{cases}$$

Corollary 3.11. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}(\tau)$. Then,*

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{6} \begin{cases} 1 & if \quad |\tau| \le 2, \\ \frac{|\tau|}{2} & if \quad |\tau| \ge 2. \end{cases}$$

Setting $\beta = 0$, $\beta = 1$, and $\tau = 1$ in Corollary 3.9 or $\tau = 1$ in Corollary 3.10 and Corollary 3.11, we obtain the following results obtained in [16] and [18], respctively.

Corollary 3.12. *If the function $f \in A$ given by (1) belong to the class $S^*_{\sinh}$ then,*

$$\left|a_3 - a_2^2\right| \le \frac{1}{2}.$$

Corollary 3.13. *If the function $f \in A$ given by (1) belong to the class $C_{\sinh}$. Then,*

$$\left|a_3 - a_2^2\right| \le \frac{1}{6}.$$

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