TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 2, 162–173
ISSN: 2587–0971

https://www.tjoscience.com

Grüss-Type Inequalities Obtained with a Generalized Fractional Integral Operator

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Abstract. In this study, new inequalities have been obtained by employing a generalized fractional integral operator which generalizes some fractional integral operators existing in the literature. In the initial step of the study, certain general properties related to Grüss type inequalities were discussed, followed by establishing a hierarchy among some fractional integral operators already existing in the literature. In the main results section, new lemmas and inequalities of Grüss type have been obtained using the aforementioned fractional integral operator.

1. Introduction

Inequalities have become a highly sought-after in the last decades due to their increasing utilization in various domains and their capacity to introduce a different perspective into science. One of the most crucial inequalities using in this field is the following which is called Grüss inequality (see [1]). Grüss proved that, for two integrable functions f and g defined on $[a, b]$:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \\ \leq \ & \frac{1}{4}(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1) \end{aligned} \quad (1)$$

holds where

$$\Phi_1 \leq f(x) \leq \Phi_2 \text{ and } \Psi_1 \leq g(x) \leq \Psi_2, \quad \Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{R}. \quad (2)$$

The well-known Grüss inequality is denoted as inequality (1) in this study. Scholars from all around the globe have shown significant interest in it. As an example, Elezovic et al.[2] formulated a series of Grüss-type inequalities associated with the Chebyshev functional within function spaces denoted as L_p with weight functions and exponents. Liu and Ngo [3] introduced an Ostrowski-Grüss type inequality on

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Received: 21 June 2024; Accepted: 2 September 2024; Published: 30 September 2024.
Keywords. Fractional integral inequalities, generalized fractional integral operator, Grüss-type fractional inequalities.
2010 *Mathematics Subject Classification*. 26A33, 26D10
Cited this article as: Aşak, Ç. & Gürbüz, M. (2024). Grüss-Type Inequalities Obtained with a Generalized Fractional Integral Operator. Turkish Journal of Science, Turkish Journal of Science, 9(2), 162–173.

time scales, consolidating analogous versions from continuous, discrete, and quantum calculus. Dragomir [4] derived precise Grüss-type inequalities for functions exhibiting bounded variation when combined with self-adjoint operators in a Hilbert space. Additionally, Dragomir [5] derived a number of Grüss-type inequalities for complex integrals, taking into account various underlying assumptions. One can see [6] to see more inequalities of Grüss type.

Definition 1.1. *([7]) The function $f : [u, v] \to \mathbb{R}$ is said to be convex, if we have*

$$f(\lambda t_1 + (1-\lambda)t_2) \le \lambda f(t_1) + (1-\lambda)f(t_2)$$

for all $t_1, t_2 \in [u, v]$ and $\lambda \in [0, 1]$.

In recent years, a multitude of generalizations, variations, and extensions of this inequality have been published, stemming from the concept of convexity ([8]-[9]). Over the past few decades, fractional calculus has witnessed significant growth in popularity and importance, primarily owing to its applications across various diverse and extensive fields. Recently, there has been a growing interest in the utilization of fractional integral operators to reinvigorate well-established integral inequalities. For example in [13] the obtained results on inequalities are demonstrated through an illustrative example, accompanied by 2D and 3D graphical representations. Also in [14], Abdeljawad et al. obtained an extension of Schweitzer's inequality to Riemann-Liouville fractional integral.

Initially, let's review the Riemann-Liouville fractional integral, as defined by [10], which will have continued significance in the context of this paper.

Definition 1.2. *Let $f \in L_1[u, v]$, the Riemann-Liouville integrals $I^{\alpha}_{u^+} f$ and $I^{\alpha}_{v^-} f$ of order α with $u \ge 0$ are defined by*

$$I^{\alpha}_{u^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-\delta)^{\alpha-1} f(\delta) d\delta, \quad x > u, \tag{3}$$

$$I^{\alpha}_{v^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (\delta-x)^{\alpha-1} f(\delta) d\delta, \quad x < v. \tag{4}$$

Here $\Gamma(\alpha)$ is the familiar Gamma function and $I^0_{u^+} f(x) = I^0_{v^-} f(x) = f(x)$.

Definition 1.3. *([10]) Let $h : [a, b] \to \mathbb{R}$ be a positive monotone and increasing function on $[a, b]$, and $h'(\delta)$ is continuously differantiable on $[a, b]$. Then $I^{\alpha}_{a^+ h} w(x)$ and $I^{\alpha}_{b^- h} w(x)$ fractional integrals of w with respect to the function h on $[a, b]$ of the order $\alpha > 0$ are defined by*

$$I^{\alpha}_{a^+ h} w(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{h'(\delta) w(\delta)}{[h(x) - h(\delta)]^{1-\alpha}} d\delta, \quad x > a \tag{5}$$

$$I^{\alpha}_{b^- h} w(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{h'(\delta) w(\delta)}{[h(\delta) - h(x)]^{1-\alpha}} d\delta, \quad x < b \tag{6}$$

Notice that for $h(x) = x$, fractional integrals (5) and (6) become Riemann-Liouville fractional integrals (3) and (4). Conversely, when $h(x) = \ln x$, fractional integrals (5) and (6) are transformed into Hadamard fractional integrals.

Definition 1.4. *([12])Let $f \in L[u, v]$, the generalized fractional integral operators $I_{u^+\phi} f$ and $I_{v^-\phi} f$ are defined by*

$$I_{u^+\phi} f(x) = \int_u^x \frac{\phi(x-\delta)}{x-\delta} f(\delta) d\delta, \quad x > u,$$

$$I_{v^-\phi} f(x) = \int_x^v \frac{\phi(\delta-x)}{\delta-x} f(\delta) d\delta, \quad x < v,$$

where a function $\phi : [0,\infty) \to [0,\infty)$ satisfies the following conditions:

$$\int_0^1 \frac{\phi(\delta)}{\delta} d\delta < \infty, \tag{7}$$

$$\frac{1}{T_1} \leq \frac{\phi(\theta)}{\phi(\xi)} \leq T_1, \quad \text{for } \frac{1}{2} \leq \frac{\theta}{\xi} \leq 2, \tag{8}$$

$$\frac{\phi(\xi)}{\xi^2} \leq T_2 \frac{\phi(\theta)}{\theta^2}, \quad \text{for } \theta \leq \xi, \tag{9}$$

$$\left| \frac{\phi(\xi)}{\xi^2} - \frac{\phi(\theta)}{\theta^2} \right| \leq T_3 \left| \xi - \theta \right| \frac{\phi(\xi)}{\xi^2}, \quad \text{for } \frac{1}{2} \leq \frac{\theta}{\xi} \leq 2, \tag{10}$$

where $T_1, T_2, T_3 > 0$ are independent of $\theta, \xi > 0$. If $\phi(\xi)\xi^\alpha$ $(\alpha > 0)$ is increasing and $\frac{\phi(\xi)}{\xi^\beta}$ $(\beta > 0)$ is decreasing, then ϕ satisfies (7) to (10).

.

Definition 1.5. *([11]) Assume that $h : [a,b] \to \mathbb{R}$ be a monotone, non-negative and increasing function on $[a,b]$. Also let its derivative function be continuously differentiable on $[a,b]$. Furthermore assume that $w : [a,b] \to \mathbb{R}$ be a positive and integrable function and the function σ satisfies (7) to (10). Under these conditions, ${}^{k}_{a^+}I^{\alpha}_{h\sigma} w(x)$ and ${}^{k}_{b^-}I^{\alpha}_{h\sigma} w(x)$ fractional integrals of w with respect to the function h on $[a,b]$ of order $\alpha > 0$ are defined by*

$${}^{k}_{a+}I^{\alpha}_{h\sigma} w(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x \frac{\sigma(h(x)-h(\delta))}{h(x)-h(\delta)} h'(\delta).(h(x)-h(\delta))^{\frac{\alpha}{k}-1} w(\delta) d\delta, \qquad x > a, \tag{11}$$

and

$${}^{k}_{b^-}I^{\alpha}_{h\sigma} w(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b \frac{\sigma(h(\delta)-h(x))}{h(\delta)-h(x)} h'(\delta).(h(\delta)-h(x))^{\frac{\alpha}{k}-1} w(\delta) d\delta, \qquad x < b. \tag{12}$$

The general operator given above reduces to usual Riemann integral, k-Riemann-Liouville fractional integral, fractional integral operator given at Definition 1.2, fractional integral operator given at Definition 1.3, fractional integral operator given at Definition 1.4, Hadamard fractional integral, etc. under some special conditions.

The motivation of this study is to obtain general inequalities which are valid for some fractional integral inequalities given above under special selections. Also obtaining results with respect to another function is motivating because they are likely to produce very valuable results.

2. Main Results

In this part new lemmas and theorems are obtained using new generalized fractional integral operator given in (11) and (12).

Theorem 2.1. *Assume that f and g are two positive integrable functions that satisfy the condition (2) on $[a,b]$. Furthermore, let u and v be two nonnegative, continuous functions on the interval $[a,b]$. Then, the following result holds:*

$$\begin{aligned} & \left| \begin{array}{c} ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma} u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma} vfg)(x) + ({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma} v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma} ufg)(x) \\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma} uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma} vg)(x) - ({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma} vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma} ug)(x) \end{array} \right| \\ \leq\ & ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma} u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma} v)(x)(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1). \end{aligned} \tag{13}$$

where $\alpha_1, \alpha_2, k_1, k_2 > 0$.

Proof. Let $Z(\tau,\rho)$ be defined by

$$Z(\tau,\rho) = (f(\tau)-f(\rho))(g(\tau)-g(\rho)), \quad \forall \tau,\rho \in [a,b]. \tag{14}$$

Multiplying both sides of (14) by $u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\upsilon(\rho)$
$\times\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}$and integrating the obtained inequality with regard to τ and ρ from a to x,respectively we can get

$$\begin{aligned}
&\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\\
&\times\upsilon(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}Z(\tau,\rho)d\tau d\rho\\
=\ &({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon fg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\
&-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon g)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon f)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x).
\end{aligned} \tag{15}$$

According to the condition(2), we have

$$\left|Z(\tau,\rho)\right| = \left|(f(\tau)-f(\rho)\right|\left|(g(\tau)-g(\rho)\right| \le (\Phi_2-\Phi_1)(\Psi_2-\Psi_1), \forall\tau,\rho\in[a,b] \tag{16}$$

Combining(15) and (16), we obtain that

$$\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon fg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon g)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon f)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\le\ &\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)\\
&\times(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\upsilon(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}\\
&\times h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|Z(\tau,\rho)\right|d\tau d\rho\\
\le\ &\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)\,(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\,\upsilon(\rho)\\
&\times\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}(\Phi_2-\Phi_1)(\Psi_2-\Psi_1)d\tau d\rho\\
=\ &({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)(\Phi_2-\Phi_1)(\Psi_2-\Psi_1).
\end{aligned} \tag{17}$$

This ends the proof. □

Theorem 2.2. *Assume that f and g are two positive integrable functions that satisfy the Lipschitz condition with constants L_1 and L_2. Furthermore, let u and υ be two nonnegative, continuous functions on the interval $[a,b]$. Then, the following result holds:*

$$\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon fg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon g)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon f)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\le\ &L_1L_2\left(\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u\right)(x)\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon i^2\right)(x)\right.\\
&\left.+\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon\right)(x)\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ui^2\right)(x)-2\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ui\right)(x)\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon i\right)(x)\right)
\end{aligned} \tag{18}$$

Proof. Based on the conditions outlined in Theorem 2.2, for $\tau,\rho\in[a,b]$ it can be derived that

$$\left|(f(\tau)-f(\rho)\right|\leq L_1\left|\tau-\rho\right| \quad and \quad \left|g(\tau)-g(\rho)\right|\leq L_2\left|\tau-\rho\right|, \tag{19}$$

which implies that

$$\left|Z(\tau,\rho)\right|=\left|f(\tau)-f(\rho)\right|\left|g(\tau)-g(\rho)\right|\leq L_1L_2(\tau-\rho)^2. \tag{20}$$

Combining (15) and (20), we get that

$$\begin{aligned}
&\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\upsilon(\rho)\\
&\times\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|Z(\tau,\rho)\right|d\tau d\rho\\
\leq\;&\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\upsilon(\rho)\\
&\times\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}L_1L_2(\tau-\rho)^2d\tau d\rho\\
=\;&L_1L_2\left(\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u\right)(x)\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon i^2\right)(x)\right.\\
&\left.+\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon\right)(x)\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ui^2\right)(x)-2\left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ui\right)(x)\left({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon i\right)(x)\right)
\end{aligned} \tag{21}$$

This end the proof of Teorem 2.2. □

Theorem 2.3. *Given that f and g are two positive integrable functions satisfying conditions $f'\in L^{\mathbf{p}}[a,b], g'\in L^{\mathbf{q}}[a,b]$ and let u and υ be two nonnegative continuous functions defined on the interval $[a,b]$, where $\mathbf{p},\mathbf{q},\mathbf{r}>1$ with $1/\mathbf{p}+1/\mathbf{p'}=1, 1/\mathbf{q}+1/\mathbf{q'}=1$ and $1/\mathbf{r}+1/\mathbf{r'}=1$. Under these assumptions, the following weighted fractional integral inequality is satisfied:*

$$\begin{aligned}
&2\left|({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right|\\
\leq\;&\|f'\|_{\mathbf{p}}\|g'\|_{\mathbf{q}}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\
&\times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p'}}+\frac{1}{\mathbf{q'}}}d\tau d\rho.
\end{aligned} \tag{22}$$

where $\alpha_1,k_{1,}>0$.

Proof. Multiplying both sides of (14) by $u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)$
$\times(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho).(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}$and integrating the given result with respect to τ and ρ from a to x, we can state that

$$\begin{aligned}
&\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\
&\times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}Z(\tau,\rho)d\tau d\rho\\
=\;&2\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right)
\end{aligned} \tag{23}$$

On the other hand from (14), we have

$$Z(\tau,\rho)=\int_{\rho}^{\tau}\int_{\rho}^{\tau} f'(\theta)g'(\vartheta)d\theta d\vartheta,\quad \forall\tau,\rho\in[a,b]\,. \tag{24}$$

By employing the Hölder inequality, we obtain

$$\left|(f(\tau)-f(\rho)\right|\le\left|\tau-\rho\right|^{\frac{1}{p'}}\left|\int_{\rho}^{\tau}\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{1}{p}}\ and\ \left|g(\tau)-g(\rho)\right|\le\left|\tau-\rho\right|^{\frac{1}{q'}}\left|\int_{\rho}^{\tau}\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{1}{q}} \tag{25}$$

Combining (24) and (25), we get

$$\left|Z(\tau,\rho)\right|\le\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_{\rho}^{\tau}\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{1}{p}}\left|\int_{\rho}^{\tau}\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{1}{q}} \tag{26}$$

According inequalities (23) and (26) we can write

$$\begin{aligned}
&2\left|({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}u)(x)({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}ufg)(x)-({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}uf)(x)({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}ug)(x)\right|\\
\le&\int_{a}^{x}\int_{a}^{x}u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\
&\times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\\
&\times\left|\int_{\rho}^{\tau}\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{1}{p}}\left|\int_{\rho}^{\tau}\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{1}{q}}d\tau d\rho\,.
\end{aligned} \tag{27}$$

Applying the double integral Hölder inequality(27), we obtain

$$\begin{aligned}
&2\left|({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}u)(x)({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}ufg)(x)-({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}uf)(x)({}_{a^+}^{k_1}I_{h\sigma}^{\alpha_1}ug)(x)\right|\\
\le&\left[\begin{array}{c}\int_{a}^{x}\int_{a}^{x}u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_{\rho}^{\tau}\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{r}{p}}d\tau d\rho\end{array}\right]^{\frac{1}{r}}\\
&\times\left[\begin{array}{c}\int_{a}^{x}\int_{a}^{x}u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_{\rho}^{\tau}\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{r'}{q}}d\tau d\rho\end{array}\right]^{\frac{1}{r'}}.
\end{aligned} \tag{28}$$

Using the following properties

$$\left|\int_{\rho}^{\tau}\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{1}{p}}\le\|f'\|_{p}^{p}\ \ and\ \ \left|\int_{\rho}^{\tau}\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{1}{q}}\le\|g'\|_{q}^{q} \tag{29}$$

then (28) can be rewritten as

$$
\begin{aligned}
&2\left|({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right| \quad (30)\\
\leq &\left[\begin{array}{l} \|f'\|^{\mathbf{r}}_{\mathbf{p}}\int\limits_a^x\int\limits_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p}'}+\frac{1}{\mathbf{q}'}}d\tau d\rho\end{array}\right]^{\frac{1}{\mathbf{r}}}\\
x&\left[\begin{array}{l} \|g'\|^{\mathbf{r}'}_{\mathbf{q}}\int\limits_a^x\int\limits_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p}'}+\frac{1}{\mathbf{q}'}}d\tau d\rho\end{array}\right]^{\frac{1}{\mathbf{r}'}}\\
= &\|f'\|_{\mathbf{p}}\|g'\|_{\mathbf{q}}\int\limits_a^x\int\limits_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\
&\times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p}'}+\frac{1}{\mathbf{q}'}}d\tau d\rho
\end{aligned}
$$

which completes the desired proof. □

Theorem 2.4. *Given that f and g are two positive integrable functions satisfying conditions $f'\in L^{\mathbf{p}}[a,b]$, $g'\in L^{\mathbf{q}}[a,b]$ and let u and v be two nonnegative continuous functions defined on the interval $[a,b]$, where $\mathbf{p},\mathbf{q},\mathbf{r}>1$ with $1/\mathbf{p}+1/\mathbf{p}'=1$, $1/\mathbf{q}+1/\mathbf{q}'=1$ and $1/\mathbf{r}+1/\mathbf{r}'=1$. Under these assumptions, the following weighted fractional integral inequality is satisfied:*

$$
\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\leq\ &\|f'\|_{\mathbf{p}}\|g'\|_{\mathbf{q}}\int\limits_a^x\int\limits_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1\frac{\alpha_1}{k_1}-1}\\
&\times v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}(h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p}'}+\frac{1}{\mathbf{q}'}})d\tau d\rho
\end{aligned}
\quad (31)
$$

where $\alpha_1,\alpha_2,k_1,k_2>0$.

Proof. Combining (17) and (26), we get

$$
\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\leq\ &\int\limits_a^x\int\limits_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h^{'}(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}\\
&\times h^{'}(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{\mathbf{p}'}+\frac{1}{\mathbf{q}'}}\left|\int_\rho^\tau\left|f'(\theta)\right|^{\mathbf{p}}d\theta\right|^{\frac{1}{\mathbf{p}}}\left|\int_\rho^\tau\left|g'(\theta)\right|^{\mathbf{q}}d\theta\right|^{\frac{1}{\mathbf{q}}}d\tau d\rho
\end{aligned}
\quad (32)
$$

By applying the Hölder inequality for double integrals to (32) we get

$$\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\leq\ &\left[\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\right.\\
&\left.\times\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_\rho^\tau\left|f'(\theta)\right|^{p}d\theta\right|^{\frac{r}{p}}d\tau d\rho\right]^{\frac{1}{r}}\\
&\times\left[\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\right.\\
&\left.\times\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}\left|\int_\rho^\tau\left|g'(\theta)\right|^{q}d\theta\right|^{\frac{r'}{q}}d\tau d\rho\right]^{\frac{1}{r'}}.
\end{aligned} \tag{33}$$

Using the following (29) can be rewritten as (33)

$$\begin{aligned}
&\left|\begin{array}{l}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\ -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right|\\
\leq\ &\left[\begin{array}{c}\|f'\|_{p}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}\\ \times h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}d\tau d\rho\end{array}\right]^{\frac{1}{r}}\\
&\times\left[\begin{array}{c}\|g'\|_{q}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}\\ \times h'(\rho).(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}}d\tau d\rho\end{array}\right]^{\frac{1}{r'}}\\
=\ &\|f'\|_{p}\|g'\|_{q}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}\\
&\times v(\rho)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}x(h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_2}{k_2}-1}\left|\tau-\rho\right|^{\frac{1}{p'}+\frac{1}{q'}})d\tau d\rho
\end{aligned} \tag{34}$$

This concludes the demonstration of Theorem 2.4. □

Lemma 2.5. *Consider an integrable function f defined on the interval $[a,b]$ that satisfies condition (2). Additionally, let u be a continuous function on $[a,b]$. Under these conditions, the following equality is valid*

$$\begin{aligned}
&({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)^2\\
=\ &\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\\
&-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2-f)(f-\Phi_1))(x).
\end{aligned} \tag{35}$$

where $\alpha_1,\alpha_2,k_1,k_2>0$.

Proof. Given that f is an integrable function that meets the condition $\Phi_1 \leq f(x) \leq \Phi_2$ we have

$$\begin{aligned} & (\Phi_2 - f(\rho))(f(\tau) - \Phi_1) + (\Phi_2 - f(\tau))(f(\rho) - \Phi_1) \\ & -(\Phi_2 - f(\tau))(f(\tau) - \Phi_1) - (\Phi_2 - f(\rho))(f(\rho) - \Phi_1) \\ = \;& f^2(\tau) + f^2(\rho) - 2f(\rho)f(\tau) \end{aligned} \tag{36}$$

for any $x, \rho, \tau \in [a, b]$. Multiplying both sides of (36) by $u(\rho)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho).(h(x) - h(\rho))^{\frac{\alpha_1}{k_1}-1}$ and integrating the obtained inequality with regard to ρ from a to x,we have

$$\begin{aligned} & (f(\tau) - \Phi_1)\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right) + (\Phi_2 - f(\tau))(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x) \\ & -\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)) - (\Phi_2 - f(\tau))(f(\tau) - \Phi_1)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2 - f)(f - \Phi_1))(x) \\ = \;& f^2(\tau)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) + ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x) - 2f(\tau)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x). \end{aligned} \tag{37}$$

Multiplying both sides of (37) by $u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau).(h(x) - h(\tau))^{\frac{\alpha_1}{k_1}-1}$ and integrating the obtained inequality with regard to τ from a to x,we have

$$\begin{aligned} & \left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x) - \Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right) \\ & + \left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x) - \Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right) \\ & - \left({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2 - f)(f - \Phi_1)\right)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2 - f)(f - \Phi_1)\right)(x) \\ = \;& ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x) + ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x) - 2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x), \end{aligned} \tag{38}$$

which gives (35). □

Theorem 2.6. *Consider two integrable functions f and g on the interval $[a, b]$ that satisfy condition (2). Additionally, let u and v be two non-negative continuous functions on $[a, b]$. Under these conditions, the following inequality is valid*

$$\begin{aligned} & \left|({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right| \\ \leq \;& \frac{1}{4}(\Phi_2 - \Phi_1)(\Psi_2 - \Psi_1)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)^2 \end{aligned} \tag{39}$$

Proof. Multiplying both sides of (14) by $u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau)$ $\times(h(x) - h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho).(h(x) - h(\rho))^{\frac{\alpha_1}{k_1}-1}$and integrating the given result with respect to τ and ρ from a to x, we can state

$$\begin{aligned} & \int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x) - h(\tau))}{h(x) - h(\tau)}h'(\tau).(h(x) - h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho) \\ & \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x) - h(\rho))}{h(x) - h(\rho)}h'(\rho)(h(x) - h(\rho))^{\frac{\alpha_1}{k_1}-1}Z(\tau, \rho)d\tau d\rho \\ = \;& 2\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x) - ({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right). \end{aligned} \tag{40}$$

Utilizing the weighted Cauchy-Schwarz integral inequality for double integrals, we can express

$$\begin{aligned}
&\left[\begin{array}{c}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau).(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}Z(\tau,\rho)d\tau d\rho\end{array}\right]^2\\
\leq&\left[\begin{array}{c}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau).(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}(f(\tau)-f(\rho))^2d\tau d\rho\end{array}\right]\\
&\times\left[\begin{array}{c}\int_a^x\int_a^x u(\tau)\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau).(h(x)-h(\tau))^{\frac{\alpha_1}{k_1}-1}u(\rho)\\ \times\frac{1}{k_1\Gamma_{k_1}(\alpha_1)}\frac{\sigma(h(x)-h(\rho))}{h(x)-h(\rho)}h'(\rho)(h(x)-h(\rho))^{\frac{\alpha_1}{k_1}-1}(g(\tau)-g(\rho))^2d\tau d\rho\end{array}\right] \quad (41)\\
=&\ 4\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x))^2\right)\\
&\times\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug^2)(x)-(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x))^2\right).
\end{aligned}$$

Since $(\Phi_2 - f(\tau))(f(\tau)- \Phi_1) \geq 0$ and $(\Psi_2 - g(\tau))(g(\tau) - \Psi_1) \geq 0$, we have

$$\begin{aligned}
({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2-f)(f-\Phi_1))(x) &\geq 0\,, \quad (42)\\
({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Psi_2-g)(g-\Psi_1))(x) &\geq 0.
\end{aligned}$$

Thus, from (42) and Lemma 2.5, we get

$$\begin{aligned}
&({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)^2 \quad (43)\\
\leq&\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)
\end{aligned}$$

$$\begin{aligned}
&({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug^2)(x)-(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x))^2 \quad (44)\\
\leq&\left(\Psi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)-\Psi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right).
\end{aligned}$$

Combining (40),(41),(43) and (44), we deduce that

$$\begin{aligned}
&\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right)^2 \quad (45)\\
\leq&\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\\
&\times\left(\Psi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)-\Psi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right).
\end{aligned}$$

Now using the elementary inequality $4xy \leq (x + y)^2$, $x, y \in \mathbb{R}$, we can state that

$$\begin{aligned}
&4(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x))(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)) \quad (46)\\
\leq&\left((\Phi_2-\Phi_1)(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)^2,\\
&4\left(\Psi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)-\Psi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right) \quad (47)\\
\leq&\left((\Psi_2-\Psi_1)(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)^2.
\end{aligned}$$

□

From (45)-(47), we obtain (39). This completes the proof of Theorem 2.6.

Lemma 2.7. *Assume f and g are positive, integrable functions defined on* $[a,b]$, *and let u and v are continuous nonnegative functions on* $[a,b]$. *Then, the inequality given below holds:*

$$\begin{aligned}&\left(\begin{array}{c}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right)^2\\
\leq\;&\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf^2)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)\right)\\
&\times\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg^2)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug^2)(x)-2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)\right).\end{aligned}\tag{48}$$

where $\alpha_1,\alpha_2,k_1,k_2>0$.

Proof. By applying the weighted Cauchy-Schwarz integral inequality for double integrals, it can be deduced from (15) that (48) is obtained. □

Lemma 2.8. *Consider f as an integrable function on* $[a,b]$, *with u and v being two nonnegative continuous functions on* $[a,b]$. *Under these conditions, the following equation is established:*

$$\begin{aligned}&({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf^2)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)\\
=\;&\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)-\Phi_1({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)\right)\\
&-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v(\Phi_2-f)(f-\Phi_1))(x)\\
&+\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\left(\Phi_2({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)\right)\\
&-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2-f)(f-\Phi_1))(x).\end{aligned}\tag{49}$$

Proof. Multiplying both sides of (37) by $v(\tau)\frac{1}{k_2\Gamma_{k_2}(\alpha_2)}\frac{\sigma(h(x)-h(\tau))}{h(x)-h(\tau)}h'(\tau).(h(x)-h(\tau))^{\frac{\alpha_2}{k_2}-1}$ and integrating the obtained equation with respect to τ from a to x, we have

$$\begin{aligned}&\left(({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)-\Phi_1({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)\right)\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\\
&-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v(\Phi_2-f)(f-\Phi_1))(x)\\
&+\left(\Phi_2({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\\
&-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2-f)(f-\Phi_1))(x)\\
=\;&({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf^2)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf^2)(x)-2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x).\end{aligned}\tag{50}$$

which gives (49) and proves the Lemma 2.8. □

Theorem 2.9. *Suppose that* f,g *are two postive integrable functions on* $[a,b]$ *satisfying the condition* (2) *and let* u,v *be two continuous nonnegative functions on* $[a,b]$ *.Under these conditions we have*

$$\begin{aligned}&\left(\begin{array}{c}({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vfg)(x)+({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ufg)(x)\\-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)\end{array}\right)^2\\
\leq\;&\left(\Phi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)\right)\left(({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)-\Phi_1({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)\right)\\
&+\left(\Phi_2({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vf)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}uf)(x)-\Phi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right)\\
&\times\left(\Psi_2({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)-{}^{k_1}_{a^+}\,I^{\alpha_1}_{h\sigma}ug)(x)\right)\left(({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)-\Psi_1({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)\right)\\
&+\left(\Psi_2({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}v)(x)-({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}vg)(x)\right)\left(({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}ug)(x)-\Psi_1({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)\right).\end{aligned}\tag{51}$$

where $\alpha_1, \alpha_2, k_1, k_2 > 0$.

Proof. Since $(\Phi_2 - f(\tau))(f(\tau) - \Phi_1) \geq 0$ and $(\Psi_2 - g(\tau))(g(\tau) - \Psi_1) \geq 0$,

$$\begin{aligned} & -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon(\Phi_2 - f)(f - \Phi_1))(x) \\ & -({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Phi_2 - f)(f - \Phi_1))(x) \\ \leq & \ 0 \end{aligned} \tag{52}$$

$$\begin{aligned} & -({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u)(x)({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon(\Psi_2 - g)(g - \Psi_1))(x) \\ & -({}^{k_2}_{a^+}I^{\alpha_2}_{h\sigma}\upsilon)(x)({}^{k_1}_{a^+}I^{\alpha_1}_{h\sigma}u(\Psi_2 - g)(g - \Psi_1))(x) \\ \leq & \ 0. \end{aligned} \tag{53}$$

□

Applying Lemma 2.8 to f and g and using (52), (53) and Lemma 2.7, we obtain (51).

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