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On Some Inequalities for Product of Different Kinds of Convex Functions

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Abstract. In this paper some new inequalities for product of different kinds of convex functions are obtained. To put forward new results, basic definitions of convex functions are considered in different ways and fairly elementary analysis is used.

1. Introduction

It is seen that, in development of mathematics and many other applied sicences, inequalities are very important. Inequalities offer so wide and effective field of study and application. To meet the application area needs of inequalities, there has been a constantly increasing interest in such an area of research. As it is known that convex functions and various forms of it are closely related with integral inequalities, many researchers have revealed many classes of convex functions in line with the growing interest in convexity theory. Interested researchers can see various generalizations and applications of convex functions in [1], [3], [4], [6]-[9] and references therein.

The function f defined from $[a, b]$ which is a subset of $\mathbb{R}$ and defined to $\mathbb{R}$, is namely convex function if it holds the inequality

$$f(\zeta r + (1-\zeta)s) \leq \zeta f(r) + (1-\zeta) f(s)$$

for all $\zeta \in [0, 1]$ and $r, s \in [a, b]$. Also f is said to be concave if $(-f)$ is convex.

E. K. Godunova and V. I. Levin established the $Q(I)$ class of function in 1985 which is defined in the following (see [2]):

Definition 1.1. *Let f be a nonnegative function defined from I to $\mathbb{R}$ belongs to the class $Q(I)$ if it satisfies the inequality*

$$f(\zeta r + (1-\zeta)s) \leq \frac{f(r)}{\zeta} + \frac{f(s)}{1-\zeta}$$

for all $\zeta \in (0, 1)$ and $r, s \in I$.

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The following class of functions are introduced by S. Varošanec in [10].

Definition 1.2. *Let h be a non-negative function defined from J (a subset of $\mathbb{R}$ and a subset of $(0,1)$) to $\mathbb{R}$ and $h \not\equiv 0$. A non-negative function f defined from I (a subset of $\mathbb{R}$) to $\mathbb{R}$ is said to be an $h-$convex function, or that f belongs to the class $SX(h,I)$, if we have*

$$f(\zeta r + (1-\zeta)s) \leq h(\zeta) f(r) + h(1-\zeta) f(s) \tag{1}$$

for all $r,s \in I$, $\zeta \in (0,1]$.

f is said to be $h-$concave, i.e. $f \in SV(h,I)$ if the inequality (1) is reversed.

On the other hand, Miheşan, introduced the following class of convexity in [5] which generalizes many types of function types.

Definition 1.3. *Let f be a function defined from $[0,b]$ to $\mathbb{R}$. If for every $\zeta \in [0,1]$, $r,s \in [0,b]$ and $(\alpha,m) \in [0,1]^2$ we have*

$$f(\zeta r + m(1-\zeta)s) \leq \zeta^{\alpha} f(r) + m(1-\zeta^{\alpha}) f(s)$$

then f is said to be $(\alpha,m)-$convex function.

Throughout we will use $M(r,s)$ and $N(r,s)$ in the meaning of "$f(r)g(r) + f(s)g(s)$" and "$f(r)g(s) + f(s)g(r)$" respectively.

In this paper, we obtained some new inequalities using some convex function classes given above. Also we used fairly elementary analysis to obtain new results.

2. Main Results

Theorem 2.1. *Let $f,g : I \to \mathbb{R}^+$ be functions and $r,s \in I$. If $f \in SX(h,I)$, $g \in Q(I)$ with $h \in L_1([0,1])$, for all $t \in (0,1)$ we have*

$$f\left(\frac{r+s}{2}\right) g\left(\frac{r+s}{2}\right) \leq 2h\left(\frac{1}{2}\right)[M(r,s) + N(r,s)] \int_0^1 \left(\frac{h(t) + h(1-t)}{t(1-t)}\right) dt.$$

Proof. Since we can write $\frac{r+s}{2} = \frac{tr+(1-t)s}{2} + \frac{(1-t)r+ts}{2}$ for all $t \in (0,1)$ and $f \in SX(h,I)$, $g \in Q(I)$ we have

$$\begin{aligned}
& f\left(\frac{r+s}{2}\right) g\left(\frac{r+s}{2}\right) \\
= \ & f\left(\frac{tr+(1-t)s}{2} + \frac{(1-t)r+ts}{2}\right) g\left(\frac{tr+(1-t)s}{2} + \frac{(1-t)r+ts}{2}\right) \\
\leq \ & 2h\left(\frac{1}{2}\right)[f(tr+(1-t)s) + f((1-t)r+ts)] \\
& \times [g(tr+(1-t)s) + g((1-t)r+ts)] \\
\leq \ & 2h\left(\frac{1}{2}\right)[h(t)f(r) + h(1-t)f(s) + h(1-t)f(r) + h(t)f(s)] \\
& \times \left[\frac{g(r)}{t} + \frac{g(s)}{1-t} + \frac{g(r)}{1-t} + \frac{g(s)}{t}\right] \\
= \ & 2h\left(\frac{1}{2}\right)[h(t)(f(r)+f(s)) + h(1-t)(f(r)+f(s))] \\
& \times \left[\frac{(1-t)(g(r)+g(s)) + t(g(r)+g(s))}{t(1-t)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= 2h\left(\frac{1}{2}\right)(f(r)+f(s))(h(t)+h(1-t))\left[\frac{g(r)+g(s)}{t(1-t)}\right] \\
&= 2h\left(\frac{1}{2}\right)(f(r)+f(s))(g(r)+g(s))\left[\frac{h(t)+h(1-t)}{t(1-t)}\right] \\
&= 2h\left(\frac{1}{2}\right)\left[\frac{h(t)+h(1-t)}{t(1-t)}\right][M(r,s)+N(r,s)].
\end{aligned}
$$

By integrating both sides respect to t over $[0,1]$, the proof is completed. □

Theorem 2.2. *Let $f, g : [r,s] \to \mathbb{R}^+$ be convex functions on $[r,s]$. Then we get*

$$
\begin{aligned}
& f\left(\frac{r+s}{2}\right)g\left(\frac{r+s}{2}\right)+\left(\frac{f(r)+f(s)}{2}\right)g\left(\frac{r+s}{2}\right)+\left(\frac{g(r)+g(s)}{2}\right)f\left(\frac{r+s}{2}\right) \\
\leq\ & \frac{3}{4}[M(r,s)+N(r,s)].
\end{aligned}
$$

Proof. Using Hermite-Hadamard inequality, for all $t \in [0,1)$ we have

$$
\begin{aligned}
t\frac{f(r)+f(s)}{2}+(1-t)f\left(\frac{r+s}{2}\right) &\leq \frac{f(r)+f(s)}{2} \\
t\frac{g(r)+g(s)}{2}+(1-t)g\left(\frac{r+s}{2}\right) &\leq \frac{g(r)+g(s)}{2}.
\end{aligned}
$$

Multiplying the above inequalities side by side we get

$$
\begin{aligned}
& t^2\frac{(f(r)+f(s))(g(r)+g(s))}{4}+(1-t)^2 f\left(\frac{r+s}{2}\right)g\left(\frac{r+s}{2}\right) \\
& +t(1-t)\left[\left(\frac{f(r)+f(s)}{2}\right)g\left(\frac{r+s}{2}\right)+\left(\frac{g(r)+g(s)}{2}\right)f\left(\frac{r+s}{2}\right)\right] \\
\leq\ & \frac{(f(r)+f(s))(g(r)+g(s))}{4}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& (1-t)^2 f\left(\frac{r+s}{2}\right)g\left(\frac{r+s}{2}\right) \\
& +t(1-t)\left[\left(\frac{f(r)+f(s)}{2}\right)g\left(\frac{r+s}{2}\right)+\left(\frac{g(r)+g(s)}{2}\right)f\left(\frac{r+s}{2}\right)\right] \\
\leq\ & \left(1-t^2\right)\frac{(f(r)+f(s))(g(r)+g(s))}{4}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
& (1-t)f\left(\frac{r+s}{2}\right)g\left(\frac{r+s}{2}\right) \\
& +t\left[\left(\frac{f(r)+f(s)}{2}\right)g\left(\frac{r+s}{2}\right)+\left(\frac{g(r)+g(s)}{2}\right)f\left(\frac{r+s}{2}\right)\right] \\
\leq\ & (1+t)\frac{(f(r)+f(s))(g(r)+g(s))}{4}. \qquad (2)
\end{aligned}
$$

By integrating both sides of (2) respect to t from 0 to 1, the desired inequality is obtained. □

Corollary 2.3. *Since f is chosen as convex function in Teorem 2.2, by using the inequality*

$$f\left(\frac{r+s}{2}\right) \leq \frac{f(r)+f(s)}{2}$$

which is a part of Hadamard's inequality we have

$$f\left(\frac{r+s}{2}\right) g\left(\frac{r+s}{2}\right) \leq \frac{1}{4}\left[M(r,s)+N(r,s)\right].$$

Theorem 2.4. *Let $f, g : [r,s] \to \mathbb{R}^{+}$ $(0 \leq r < s)$ be (α_1, m) −convex and (α_2, m) −convex functions on $[r,s]$ respectively. If $f, g \in L_1[r,s]$, for all $x \in [r,s]$ and $(\alpha_1, m), (\alpha_2, m) \in (0,1]^2$ we have*

$$\begin{aligned}
& \frac{1}{s-r}\int_r^s f(x)\, g(r+s-x)\, dx \\
\leq\ & \beta(\alpha_1+1, \alpha_2+1)\left((fg)(r)+m^2 (fg)\left(\frac{s}{m}\right)\right) \\
& +\frac{m}{\alpha_1+\alpha_2+1}\left(f(r)\, g\left(\frac{s}{m}\right)+f\left(\frac{s}{m}\right) g(r)\right).
\end{aligned}$$

Proof. Since f and g are (α_1, m) −convex and (α_2, m) −convex on $[r,s]$ respectively, for all $t \in [0,1]$ we can write

$$\begin{aligned}
f(tr+(1-t)s) &\leq t^{\alpha_1} f(r)+m(1-t^{\alpha_1}) f\left(\frac{s}{m}\right) \\
g((1-t)r+ts) &\leq (1-t^{\alpha_2})\, g(r)+m t^{\alpha_2} g\left(\frac{s}{m}\right).
\end{aligned}$$

By multiplying the above inequalities side by side we get

$$\begin{aligned}
& f(tr+(1-t)s)\, g((1-t)r+ts) \\
\leq\ & t^{\alpha_1}(1-t^{\alpha_2})(fg)(r)+m^2(1-t^{\alpha_1})\, t^{\alpha_2}(fg)\left(\frac{s}{m}\right) \\
& +m\left[t^{\alpha_1+\alpha_2} f(r)\, g\left(\frac{s}{m}\right)+(1-t^{\alpha_1})(1-t^{\alpha_2})\, f\left(\frac{s}{m}\right) g(r)\right].
\end{aligned}$$

Moreover, for all $t_1, t_2 \in [0,1]$ and $\alpha \in (0,1]$ we have

$$\left|t_1^{\alpha}-t_2^{\alpha}\right| \leq |t_1-t_2|^{\alpha}.$$

So it is clear that

$$1-t^{\alpha_1}=1^{\alpha_1}-t^{\alpha_1} \leq (1-t)^{\alpha_1}$$

and

$$1-t^{\alpha_2}=1^{\alpha_2}-t^{\alpha_2} \leq (1-t)^{\alpha_2}.$$

With the help of above inequalities we get

$$\begin{aligned}
& f(tr+(1-t)s)\, g((1-t)r+ts) \\
\leq\ & t^{\alpha_1}(1-t)^{\alpha_2}(fg)(r)+m^2(1-t)^{\alpha_1} t^{\alpha_2}(fg)\left(\frac{s}{m}\right) \\
& +m\left[t^{\alpha_1+\alpha_2} f(r)\, g\left(\frac{s}{m}\right)+(1-t)^{\alpha_1}(1-t)^{\alpha_2} f\left(\frac{s}{m}\right) g(r)\right] \\
=\ & t^{\alpha_1}(1-t)^{\alpha_2}(fg)(r)+m^2(1-t)^{\alpha_1} t^{\alpha_2}(fg)\left(\frac{s}{m}\right) \\
& +m\left[t^{\alpha_1+\alpha_2} f(r)\, g\left(\frac{s}{m}\right)+(1-t)^{\alpha_1+\alpha_2} f\left(\frac{s}{m}\right) g(r)\right]. \qquad (3)
\end{aligned}$$

Integrating both sides of (3) respect to t over $[0,1]$ we have

$$\begin{aligned}
& \int_0^1 f(tr+(1-t)s)\, g((1-t)r+ts)\, dt \\
\leq\ & \beta(\alpha_1+1,\alpha_2+1)\left((fg)(r)+m^2(fg)\left(\frac{s}{m}\right)\right) \\
& +\frac{m}{\alpha_1+\alpha_2+1}\left(f(r)\, g\left(\frac{s}{m}\right)+f\left(\frac{s}{m}\right)g(r)\right).
\end{aligned}$$

where

$$\beta(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}dt, \qquad r,s>0$$

and

$$\beta(r,s)=\beta(s,r).$$

So the proof is completed. □

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