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Some results of generalized k-fractional integral operator with k-Bessel function

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Abstract. In this research paper, we develop the generalized fractional k-integral operators (gFkIO) involving Appell k-function as its kernel, and investigate (gFkIO) with the composition of Bessel k-function of first kind (BkF-I). We shall obtain results by applying Sagio fractional integral k-operators (SFIkO) and Riemann Liouville fractional integral k-operators (RLFIkO) in which Gauss Hypergeometric k-function (GHkF) acting as a kernel in the left and right sense with product of power k-function and Bessel k-function of first kind (BkF-I) and results will be establish in the terms of generalized Wright Hypergeometric k-function (gWHkF).

1. Introduction

Fractional calculus is the field of mathematical analysis, which deals with the investigation and applications of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integrals and derivatives. It has gained significance and recognition over the last four decades, specially because of its enormous capacity of tested programs in diverse seemingly expanded fields of science, applied mathematics and engineering [1–3]. We proposed a unified approach to the special functions of fractional calculus and our approach is based on the usage of generalized fractional calculus operators. Diaz and Pariguan [4, 5] paved the way for extensions of fractional calculus when they introduced the gamma k-function, beta k-function and hypergeometric k-functions based on Pochhammer's k-symbols [6, 7] and proved a number of their properties.

Different additions of numerous fractional integral operators and their properties have been investigated by many authors [8–10]. Many applications and special cases of generalized fractional integral operators are the recurring appearance of compositions of classical Riemann Liouville and Erdelyi Kober fractional operators in various problems of applied analysis and several properties of this operator can be located in [11, 12]. Many authors added a family of fractional integral operators with the Appell function F_3 in their kernel and extension of many acknowledged formulas given [13–15]. A distinct account of such operators along with their properties and applications had been considered [16–21].

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Definition 1.1. *The generalized fractional integral k-operator defined for α, α', β, β',$\gamma \in \mathbb{C}$ and $y > 0$, $\Re(\gamma) > 0$ and k is any real number respectively*

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}f)(y) = \frac{y^{\frac{-\alpha}{k}}}{k\Gamma_k(\gamma)}\int_0^y (y-t)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha'}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{y};1-\frac{y}{t})f(t)dt \tag{1}$$

and

$$(I_{k,y^-}^{\alpha,\alpha',\beta,\beta',\gamma}f)(y) = \frac{y^{\frac{-\alpha'}{k}}}{k\Gamma_k(\gamma)}\int_y^\infty (t-y)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{y}{t};1-\frac{t}{y})f(t)dt. \tag{2}$$

Definition 1.2. *[22] The left and right sided Sagio fractional integral k-operator defined for $\alpha,\beta,\gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $y > 0$ and k is any real number respectively as*

$$(I_{k,0^+}^{\alpha,\beta,\gamma}f)(y) = \frac{y^{\frac{-\alpha-\beta}{k}}}{k\Gamma_k(\alpha)}\int_0^y (y-t)^{\frac{\alpha}{k}-1}{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{y})f(t)dt \tag{3}$$

and

$$(I_{k,y^-}^{\alpha,\beta,\gamma}f)(y) = \frac{1}{k\Gamma_k(\alpha)}\int_y^\infty (t-y)^{\frac{\alpha}{k}-1}t^{\frac{-\alpha-\beta}{k}}{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{y}{t})f(t)dt. \tag{4}$$

Definition 1.3. *[22] The left and right sided Riemann Liouville fractional integral k-operator defined for $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, $y > 0$ and k is any positive real number respectively*

$$(I_{k,0^+}^{\alpha}f)(y) = \frac{1}{k\Gamma_k(\alpha)}\int_0^y (y-t)^{\frac{\alpha}{k}-1}f(t)dt \tag{5}$$

and

$$(I_{k,0^-}^{\alpha}f)(y) = \frac{1}{k\Gamma_k(\alpha)}\int_y^\infty (t-y)^{\frac{\alpha}{k}-1}f(t)dt. \tag{6}$$

Definition 1.4. *The k-beta function [24], defined for $\Re(l) > 0$, $\Re(h) > 0$, as*

$$\beta_k(l,h) = \frac{1}{k}\int_0^1 s^{\frac{l}{k}-1}(1-s)^{\frac{h}{k}-1}ds, \tag{7}$$

$$\text{so that} \qquad \beta_k(l,h) = \frac{1}{k}\beta\Big(\frac{l}{k},\frac{h}{k}\Big) \qquad \text{and} \qquad \beta_k(l,h) = \frac{\Gamma_k(l)\Gamma_k(h)}{\Gamma_k(l+h)}, \tag{8}$$

where $\Gamma_k(l)$, $\Gamma_k(h)$ and $\Gamma_k(l+h)$ are gamma k-functions.

Definition 1.5. *The gamma k-function [24], defined for $\Re(t) > 0$, $k > 0$, $t \in \mathbb{C}$ as*

$$\Gamma_k(t) = \int_0^\infty s^{t-1}e^{\frac{-s^k}{k}}ds, \tag{9}$$

$$\text{so that} \qquad \Gamma_k(z+k) = z\Gamma_k(z) \quad \text{and} \quad \Gamma_k(\gamma) = (k)^{\frac{\gamma}{k}-1}\Gamma(\frac{\gamma}{k}). \tag{10}$$

Definition 1.6. *The Pochhammer's k-symbol for $k > 0$ [5], defined as*

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k) & \text{for } n \geq 1 \\ 1 & \text{for } n = 0, \alpha \neq 0, \end{cases} \tag{11}$$

$$\text{So that} \qquad (\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)} \qquad \text{and} \qquad \frac{\Gamma_k(\alpha)}{\Gamma_k(\alpha-n)} = (-1)^n(k-\alpha)_{n,k} \tag{12}$$

where $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$.

Definition 1.7. *The Hypergeometric k-function defined for* $\forall\ \alpha', \beta', \eta' \in \mathbb{C},\ \eta' \neq 0, -1, -2, -3, \cdots,\ |t| < 1,$ *as*

$$ {}_2F_{1,k}((\alpha', k), (\beta', k); (\eta', k); t) = \sum_{m=0}^{\infty} \frac{(\alpha')_{m,k} (\beta')_{m,k}}{(\eta')_{m,k}} \frac{t^m}{m!}, \qquad k > 0, \tag{13} $$

$$ {}_2F_{1,k}((a, k), (b, k); (c, k); 1) = \frac{\Gamma_k(c)\Gamma_k(c - a - b)}{\Gamma_k(c - a)\Gamma_k(c - b)}, \tag{14} $$

where $\Gamma_k(c)$, $\Gamma_k(c - a - b)$, $\Gamma_k(c - a)$ *and* $\Gamma_k(c - b)$ *are gamma k-functions.*

Definition 1.8. *The generalized Wright Hypergeometric k-function [25], defined by the series as*

$$ {}_l\psi_h^k(t) = {}_l\Psi_h^k \left[\begin{array}{c} (c_i, \alpha'_i)_{1,l} \\ (d_j, \beta'_j)_{1,h} \end{array} \middle| t \right] \equiv \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{l} \Gamma_k(c_i + \alpha'_i m) t^m}{\prod_{j=1}^{h} \Gamma_k(d_j + \beta'_j m) m!}, \tag{15} $$

where $k \in \Re^+$, $t \in \mathbb{C}$, $c_i, d_j \in \mathbb{C}$, *and* $\alpha'_i, \beta'_j \in \Re$ $(i = 1, 2, \cdots, l;\ j = 1, 2, \cdots, h)$.

Definition 1.9. *The Bessel k-function of first kind* $W_{\upsilon,c}^k(t)$ *[12], defined for* $t \in \mathbb{C}$ *and* $\upsilon \in \mathbb{C}$ *by*

$$ W_{\upsilon,c}^k(t) = \sum_{p=0}^{\infty} \frac{(-c)^p (\frac{t}{2})^{\frac{\upsilon}{k} + 2p}}{\Gamma_k(\upsilon + pk + k) p!}, \quad k > 0, c \in \Re. \tag{16} $$

We use the following notation in our results

$$ \mathcal{E}^{p,k} = \sum_{p=0}^{\infty} \frac{(-c)^p (\frac{1}{2})^{\frac{\upsilon}{k} + 2p}}{\Gamma_k(\upsilon + pk + k) p!}, \qquad as \qquad W_{\upsilon,c}^k(t) = \mathcal{E}^{p,k} (t)^{\frac{\upsilon}{k} + 2p}. \tag{17} $$

2. Left sided integral *k*-operators with Bessel *k*-function

In this section, we derive the fundamental results for left sided Sagio fractional integral *k*-operator in which Gauss hypergeometric *k*-function using as a kernel with the composition of power function and Bessel *k*-function, and also discuss the left sided Riemann Liouville fractional integral *k*-operator. The following theorems are needed to prove our main results.

Theorem 2.1. *Let* α, α', β, β', γ, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ *and* $x > 0$ *be such that* $\Re(\upsilon) > -1, \Re(\gamma) > 0$ *and* $\Re(\frac{\sigma+\upsilon}{k}) > max[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')]$, *then there holds the following relation:*

$$ (I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma} [t^{\frac{\sigma}{k} - 1} W_{\upsilon,c}^k(t)])(x) = x^{\frac{1}{k}(\sigma + \upsilon + \gamma - \alpha - \alpha') - 1} (2k)^{\frac{-\upsilon}{k}} $$

$$ {}_3\psi_4^k \left[\begin{array}{c} (\sigma + \upsilon, 2)(\sigma + \upsilon + \gamma - \alpha - \alpha' - \beta, 2)(\sigma + \upsilon + \beta' k - \alpha', 2) \\ (\upsilon + 1, 1)(\sigma + \upsilon + \gamma - \alpha - \alpha', 2)(\sigma + \upsilon + \gamma - \alpha' - \beta, 2)(\sigma + \upsilon + \beta' k, 2) \end{array} \middle| -\frac{cx^2}{4k} \right]. $$

Proof. Consider the generalized *k*-fractional integral (1) with the product of power function and Bessel

k-function of first kind (16), we have

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= \sum_{p=0}^{\infty} \frac{(-c)^p(\frac{1}{2})^{2p+\frac{v}{k}}}{\Gamma_k(pk+v+k)p!}\Big[\frac{x^{-\frac{\alpha}{k}}}{k\Gamma_k(\gamma)}\int_0^x (x-t)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha'}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{t}{x};1-\frac{x}{t})t^{\frac{\sigma+v}{k}+2p-1}\Big]dt$$

$$= \mathcal{E}^{p,k}\Big[\frac{x^{-\frac{\alpha}{k}}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^x (x-t)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha'}{k}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}(1-\frac{t}{x})^m(1-\frac{x}{t})^n t^{\frac{\sigma+v}{k}+2p-1}\Big]dt$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\gamma-\alpha}{k}-1}}{k\Gamma_k(\gamma)}\int_0^x (1-\frac{t}{x})^{\frac{\gamma}{k}+m-1}(1-\frac{x}{t})^n t^{\frac{\sigma+v-\alpha'}{k}+2p-1}\Big]dt. \quad (18)$$

By putting $u=\frac{t}{x} \Rightarrow xdu=dt$, if $t=0 \Rightarrow u=0$, if $t=x \Rightarrow u=1$ in equation (18), we get

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\gamma-\alpha}{k}-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^1 (1-u)^{\frac{\gamma}{k}+m-1}(1-\frac{1}{u})^n(xu)^{\frac{\sigma+v-\alpha'}{k}+2p-1}\Big]xdu$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+v-\alpha'}{k}+2p-n-1}(1-u)^{\frac{\gamma}{k}+m+n-1}\Big]du.$$

Using equations (7) and equation (13) in equation (19), we have

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{\Gamma_k(\gamma)(\gamma)_{m+n,k}\ m!n!}\Big[x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\beta_k(\sigma+v-\alpha'+2pk-nk,\gamma+mk+nk)\Big].$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{\Gamma_k(\sigma+v-\alpha'+2pk-nk)\Gamma_k(\gamma+mk+nk)}{\Gamma_k(\sigma+v-\alpha'+2pk-nk+\gamma+mk+nk)}\Big]. \quad (19)$$

By using equation (12) in equation (19), we obtain

$$(I_{k,0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^{k}(t)])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}}{\Gamma_k(\gamma)}\Big[\frac{\Gamma_k(\sigma+v-\alpha'+2pk-nk)\Gamma_k(\gamma)(\gamma)_{m+n,k}}{\Gamma_k(\sigma+v-\alpha'+2pk+\gamma)(\sigma+v-\alpha'+2pk+\gamma)_{m,k}}\Big]$$

$$= x^{\frac{\sigma+v+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta)_{m,k}(1)^m}{(\sigma+v-\alpha'+2pk+\gamma)_{m,k}m!}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(\sigma+v-\alpha'+2pk-nk)}{\Gamma_k(\sigma+v-\alpha'+2pk+\gamma)}. \quad (20)$$

By using equation (14) in equation (20), we get

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x)$$

$$= x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)\Gamma_k(\sigma+\upsilon-\alpha'+2pk-nk)}{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)}. \quad (21)$$

Now we use equation (12) in equation (21), we have

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x)$$

$$= \sum_{n=0}^{\infty}\frac{x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}(\alpha')_{n,k}(\beta')_{n,k}(-1)^n}{(k-(\sigma+\upsilon-\alpha'+2pk))_{n,k}n!}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+\upsilon+\gamma}{k}-1}\mathcal{E}^{p,k}\Gamma_k(k-\sigma-\upsilon+\alpha'-2pk)\Gamma_k(k-\sigma-\upsilon-2pk-\beta')\Gamma_k(\sigma+\upsilon-\alpha'+2pk)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)}{x^{\frac{\alpha+\alpha'}{k}-2p}\Gamma_k(k-\sigma-\upsilon+\alpha'-2pk-\beta')\Gamma_k(k-\sigma-\upsilon-2pk)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+\upsilon+\gamma}{k}}\mathcal{E}^{p,k}(\sigma+\upsilon-\alpha'+2pk)_{\beta',k}}{x^{\frac{\alpha+\alpha'}{k}+1-2p}(\sigma+\upsilon+2pk)_{\beta',k}}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)}. \quad (22)$$

Using the equation (12) in equation (22), we obtain

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x)$$

$$= x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+2pk+\beta'k)\Gamma_k(\sigma+\upsilon+2pk)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)}{\Gamma_k(\sigma+\upsilon+2pk+\beta'k)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)}$$

$$= \frac{x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}+2p-1}\mathcal{E}^{p,k}\Gamma_k(\sigma+\upsilon+2pk)}{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha)}\frac{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\alpha-\beta)\Gamma_k(\sigma+\upsilon-\alpha'+2pk+\beta'k)}{\Gamma_k(\sigma+\upsilon-\alpha'+\gamma+2pk-\beta)\Gamma_k(\sigma+\upsilon+2pk+\beta'k)}. \quad (23)$$

By using equations (17) and equation (10) in equation (23), we get

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon-\alpha-\alpha'+\gamma}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{p=0}^{\infty}\Bigg[\frac{\Gamma_k(\sigma+\upsilon+2pk)}{\Gamma_k(\upsilon+p+1)\Gamma_k(\sigma+\upsilon+\beta'k+2pk)}$$

$$\times\frac{\Gamma_k(\sigma+\upsilon+\gamma-\alpha-\alpha'-\beta+2pk)\Gamma_k(\sigma+\upsilon+\beta'k-\alpha'+2pk)}{\Gamma_k(\sigma+\upsilon+\gamma-\alpha-\alpha'+2pk)\Gamma_k(\sigma+\upsilon+\gamma-\alpha'-\beta+2pk)}\Bigg]\frac{(\frac{-cx^2}{4k})^p}{p!}. \quad (24)$$

By using equation (15) in equation (24), and get the final result

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-\upsilon}{k}}$$

$$ {}_3\psi_4^k\left[\begin{matrix}(\sigma+\upsilon,2)(\sigma+\upsilon+\gamma-\alpha-\alpha'-\beta,2)(\sigma+\upsilon+\beta'k-\alpha',2)\\(\upsilon+1,1)(\sigma+\upsilon+\gamma-\alpha-\alpha',2)(\sigma+\upsilon+\gamma-\alpha'-\beta,2)(\sigma+\upsilon+\beta'k,2)\end{matrix}\;\middle|\; -\frac{cx^2}{4k}\right].$$

□

Corollary 2.2. *Taking $k = 1, c = 1$ in Theorem* (2.1), *we get*

$$(I_{0^+}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\sigma-1}J_v(t)])(x) = x^{\sigma+v+\gamma-\alpha-\alpha'-1}(2)^{-v}$$

$$ {}_3\psi_4\left[\begin{matrix} (\sigma+v,2)(\sigma+v+\gamma-\alpha-\alpha'-\beta,2)(\sigma+v+\beta'-\alpha',2) \\ (v+1,1)(\sigma+v+\gamma-\alpha-\alpha',2)(\sigma+v+\gamma-\alpha'-\beta,2)(\sigma+v+\beta',2)\end{matrix}\middle| -\frac{x^2}{4}\right].$$

Theorem 2.3. *Let $\alpha,\beta,\gamma,v,\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ and $x > 0$ be such that $\Re(v) > -1, \Re(\alpha) > 0$ and $\Re(\frac{\sigma+v}{k}) > max[0, \Re(\beta-\gamma)]$, then the following results holds true:*

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) = \frac{x^{\frac{\sigma+v-\beta}{k}-1}}{(2k)^{\frac{v}{k}}}\, {}_2\psi_3^k\left[\begin{matrix}(\sigma+v,2),(\sigma+v-\beta+\gamma,2)\\ (v+1,1),(\sigma+v-\beta,2),(\sigma+v+\alpha+\gamma,2)\end{matrix}\middle| -\frac{cx^2}{4k}\right].$$

Proof. Consider the left sided Saigo fractional k-integral operator (3) with the product of power function and Bessel k-function of first kind (16), we have

$$\begin{aligned}
&(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x)\\
&= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{v}{k}}}{\Gamma_k(nk+v+k)n!}\Big[\frac{x^{\frac{-\alpha-\beta}{k}}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^x (x-t)^{\frac{\alpha}{k}-1}{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x})t^{\frac{\sigma+v}{k}+2n-1}\Big]dt\\
&= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{-\alpha-\beta}{k}}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_0^x (x-t)^{\frac{\alpha}{k}-1}(1-\frac{t}{x})^m\ t^{\frac{\sigma+v}{k}+2n-1}\Big]dt\\
&= \mathcal{E}^{n,k}\frac{x^{\frac{-\alpha-\beta+\alpha}{k}-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{1}{k}\int_0^x (1-\frac{t}{x})^{\frac{\alpha}{k}+m-1}t^{\frac{\sigma+v}{k}+2n-1}\Big]dt.
\end{aligned}\tag{25}$$

By putting $u = \frac{t}{x} \Rightarrow xdu = dt$ if $t = 0 \Rightarrow u = 0$ if $t = x \Rightarrow u = 1$ in (25), we have

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) = \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+v}{k}+2n-1}(1-u)^{\frac{\alpha}{k}+m-1}\Big]du.\tag{26}$$

Using equation (7) in equation (26), we obtain

$$\begin{aligned}
(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\beta_k(\sigma+v+2nk,\ \alpha+mk)\Big]\\
&= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)\Gamma_k(\alpha+mk)}{\Gamma_k(\sigma+v+2nk+\alpha+mk)}\Big].
\end{aligned}\tag{27}$$

Using equation (12) in equation (27), we have

$$\begin{aligned}
(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{v,c}^k(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+v-\beta}{k}+2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)\Gamma_k(\alpha)(\alpha)_{m,k}}{\Gamma_k(\sigma+v+\alpha+2nk)(\sigma+v+\alpha+2nk)_{m,k}}\Big]\\
&= \mathcal{E}^{n,k}x^{\frac{\sigma+v-\beta}{k}+2n-1}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}(1)^m}{(\sigma+v+\alpha+2nk)_{m,k}m!}\Big[\frac{\Gamma_k(\sigma+v+2nk)}{\Gamma_k(\sigma+v+\alpha+2nk)}\Big].
\end{aligned}\tag{28}$$

By using equation (13) in equation (28), we have

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) = \mathcal{E}^{n,k}x^{\frac{\sigma+\upsilon-\beta}{k}+2n-1}\frac{\Gamma_k(\sigma+\upsilon+2nk)\Gamma_k(\sigma+\upsilon+2nk-\beta+\gamma)}{\Gamma_k(\sigma+\upsilon+2nk-\beta)\Gamma_k(\sigma+\upsilon+\alpha+2nk+\gamma)}. \tag{29}$$

By using equations (17) and equation (10) in equation (29), we attain

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) = \frac{x^{\frac{\sigma+\upsilon-\beta}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma(\frac{\upsilon}{k}+1+n)}\frac{\Gamma_k(\sigma+\upsilon-\beta+\gamma+2nk)}{\Gamma(\sigma+\upsilon-\beta+2nk)\Gamma(\sigma+\upsilon+\alpha+\gamma+2nk)}\Big]\frac{(\frac{-cx^2}{4k})^n}{n!}. \tag{30}$$

By using equation (15) in equation (30), we get the final result

$$(I_{k,0^+}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) = \frac{x^{\frac{\sigma+\upsilon-\beta}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\,{}_2\psi_3^k\Big[\begin{array}{c}(\sigma+\upsilon,2),(\sigma+\upsilon-\beta+\gamma,2)\\(\frac{\upsilon}{k}+1,1),(\sigma+\upsilon-\beta,2),(\sigma+\upsilon+\alpha+\gamma,2)\end{array}\Big| -\frac{cx^2}{4k}\Big].$$

□

Theorem 2.4. *Let $\alpha,\upsilon,\sigma \in \mathbb{C}$, $k>0$, $c \in \Re$ and $x>0$ be such that $\Re(\upsilon) > -1 and \Re(\alpha) > 0$, then there holds following formula:*

$$(I_{k,0^+}^{\alpha}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) = \frac{x^{\frac{\sigma+\upsilon+\alpha}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\,{}_1\psi_2^k\Big[\begin{array}{c}(\sigma+\upsilon,2)\\(\frac{\upsilon}{k}+1,1),(\sigma+\upsilon+\alpha,2)\end{array}\Big| -\frac{cx^2}{4k}\Big].$$

Proof. Consider the left sided Riemann Liouville k-fractional integral operator (5) with the product of power function and Bessel k-function of first kind (16), we have

$$\begin{aligned}(I_{k,0^+}^{\alpha}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) &= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_0^x(x-t)^{\frac{\alpha}{k}-1}t^{\frac{\sigma+\upsilon}{k}+2n-1}\Big]dt\\ &= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^x(1-\frac{t}{x})^{\frac{\alpha}{k}-1}t^{\frac{\sigma+\upsilon}{k}+2n-1}\Big]dt.\end{aligned} \tag{31}$$

By putting $u=\frac{t}{x} \Rightarrow xdu = dt$, if $t=0 \Rightarrow u=0$, if $t=x \Rightarrow u=1$ in equation (31), we get

$$(I_{k,0^+}^{\alpha}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) = \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}}{\Gamma_k(\alpha)}\Big[\frac{1}{k}\int_0^1 u^{\frac{\sigma+\upsilon+2nk}{k}-1}(1-u)^{\frac{\alpha}{k}-1}\Big]du. \tag{32}$$

By using equation (7) in equation (32), we attain

$$\begin{aligned}(I_{k,0^+}^{\alpha}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) &= \mathcal{E}^{n,k}\frac{x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}}{\Gamma_k(\alpha)}\beta_k(\sigma+\upsilon+2nk,\alpha)\\ &= x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}\mathcal{E}^{n,k}\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma_k(\sigma+\upsilon+\alpha+2nk)}.\end{aligned} \tag{33}$$

By using the equations (10) and equation (17) in equation (33), we have

$$\begin{aligned}(I_{k,0^+}^{\alpha}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^k(t)])(x) &= x^{\frac{\sigma+\upsilon+\alpha}{k}+2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{\frac{\upsilon}{k}+n+1-1}\Gamma(\frac{\upsilon}{k}+n+1)}\frac{\Gamma_k(\sigma+\upsilon+2nk)}{\Gamma_k(\sigma+\upsilon+\alpha+2nk)}\\ &= \frac{x^{\frac{\sigma+\upsilon}{k}+\alpha-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma(\sigma+\upsilon+2nk)}{\Gamma(\frac{\upsilon}{k}+1,1)\Gamma(\sigma+\upsilon+\alpha+2nk)}\frac{(\frac{-cx^2}{4k})^n}{n!}.\end{aligned} \tag{34}$$

By using equation (15) in equation (34), we get the final result

$$(I^{\alpha}_{k,0^+}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(t)])(x) = \frac{x^{\frac{\sigma+\upsilon+\alpha}{k}-1}}{(2k)^{\frac{\upsilon}{k}}}\ {}_1\psi^k_2\left[\begin{array}{c}(\sigma+\upsilon,2)\\ (\frac{\upsilon}{k}+1,1),(\sigma+\upsilon+\alpha,2)\end{array}\Big| -\frac{cx^2}{4k}\right].$$

□

3. Right sided fractional *k*-operators with Bessel *k*-function

In this section, we elaborate the right sided Sagio fractional integral *k*-operator in which hypergeometric *k*-function using as a kernel with Bessel *k*-function, and also derived Riemann Liouville fractional *k*-operator in the form of theorems.

Theorem 3.1. *Let α, α', β, β', γ, υ, $\sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ and $x > 0$ be such that $\Re(\upsilon) > -1, \Re(\gamma) > 0$ and $\Re(\frac{\sigma+\upsilon}{k}) > max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$. then there holds the following relation:*

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-\upsilon}{k}}$$

$${}_3\psi^k_4\left[\begin{array}{c}(k-\sigma+\upsilon-\beta,2),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k,2),(k-\sigma+\upsilon+\alpha+\beta'-\gamma,2)\\ (\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k+\beta',2),(k-\sigma+\upsilon-\gamma+\alpha-\beta,2),(k-\sigma+\upsilon,2)\end{array}\Big| -\frac{cx^2}{4k}\right].$$

Proof. Consider the right sided generalized fractional *k*-operator (2) with the composition of power function and Bessel *k*-function of first kind (16), we have

$$\begin{aligned}
&(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x)\\
&=\sum_{p=0}^{\infty}\frac{(-c)^p(\frac{1}{2})^{2p+\frac{\upsilon}{k}}}{\Gamma_k(pk+\upsilon+k)p!}\Big[\frac{x^{-\frac{\alpha'}{k}}}{\Gamma_k(\gamma)}\times\frac{1}{k}\int_x^{\infty}(t-x)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha}{k}}F_{3,k}(\alpha,\alpha',\beta,\beta',\gamma;1-\frac{x}{t};1-\frac{t}{x})t^{\frac{\sigma-\upsilon}{k}-2p-1}\Big]dt\\
&=\mathcal{E}^{p,k}\Big[\frac{x^{-\frac{\alpha'}{k}}}{k\Gamma_k(\gamma)}\int_x^{\infty}(t-x)^{\frac{\gamma}{k}-1}t^{-\frac{\alpha}{k}}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}(1-\frac{x}{t})^m(1-\frac{t}{x})^n t^{\frac{\sigma-\upsilon}{k}-2p-1}\Big]dt\\
&=\mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{-\frac{\alpha'}{k}}}{\Gamma_k(\gamma)}\frac{1}{k}\int_x^{\infty}(1-\frac{x}{t})^{\frac{\gamma}{k}+m-1}(1-\frac{t}{x})^n t^{\frac{\sigma-\upsilon-\alpha+\gamma}{k}-2p-2}\Big]dt.
\end{aligned}\tag{35}$$

By putting $u=\frac{x}{t}\Rightarrow -xu^2du = dt$, if $t=\infty\Rightarrow u=0$, if $t=x\Rightarrow u=1$ in equation (35), we have

$$(I^{\alpha,\alpha',\beta,\beta',\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x)\tag{36}$$

$$=\mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{-\alpha'}{k}-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_1^{0}(1-u)^{\frac{\gamma}{k}+m-1}(1-\frac{1}{u})^n(xu^{-1})^{\frac{\sigma-\upsilon-\alpha+\gamma}{k}-2p-2}\Big](-xu^{-2})du$$

$$=\mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\frac{1}{k}\int_0^1 u^{\frac{k-\sigma+\upsilon+\alpha-\gamma+2pk-nk}{k}-1}(1-u)^{\frac{\gamma+mk+nk}{k}-1}\Big]du.\tag{37}$$

By using equation (7) and equation (8) in equation (37), we obtain

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\Big[\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\beta_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk,\gamma+mk+nk)\Big]$$

$$= \mathcal{E}^{p,k}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}\ m!n!}\frac{x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}}{\Gamma_k(\gamma)}\Big[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)\Gamma_k(\gamma+mk+nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk+\gamma+mk+nk)}\Big]. \quad (38)$$

By using the equation (12) in equation (38), we have

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta')_{m,k}(\beta)_{n,k}}{\Gamma_k(\gamma)(\gamma)_{m+n,k}\ m!n!}\Big[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)\Gamma_k(\gamma)(\gamma)_{m+n,k}}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk)(k-\sigma+\upsilon+\alpha+2pk)_{m,k}}\Big].$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{m=0}^{\infty}\frac{(\alpha)_{m,k}(\beta)_{m,k}(1)^m}{(k-\sigma+\upsilon+\alpha+2pk)_{m,k}\ m!}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\Big[\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk)}\Big]. \quad (39)$$

By using the equation (14) in equation (39), we get

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}}{n!}\frac{\Gamma_k(k-\sigma+\upsilon+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk-nk)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+2pk)}. \quad (40)$$

Now, we use the equation (12) in equation (40), we have

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\sum_{n=0}^{\infty}\frac{(\alpha')_{n,k}(\beta')_{n,k}(-1)^n}{(\sigma-\upsilon-\alpha+\gamma-2pk)_{n,k}n!}\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon+2pk-\beta)}{\Gamma_k(k-\sigma+\upsilon+\alpha+2pk-\beta)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma-\upsilon-\alpha+\gamma-\alpha'}{k}-2p-1}\frac{\Gamma_k(\sigma-\upsilon+\gamma-\alpha-2pk)\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\alpha'-\beta'-2pk)}{\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\alpha'-2pk)\Gamma_k(\sigma-\upsilon+\gamma-\alpha-\beta'-2pk)}$$
$$\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}-2p-1}\frac{(k-\sigma+\upsilon-\gamma+\alpha+2pk)_{\alpha',k}}{(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)_{\alpha',k}}\frac{\Gamma_k(k-\sigma+\upsilon+\alpha-\gamma+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)}$$

$$= \mathcal{E}^{p,k}x^{\frac{\sigma+\upsilon+\gamma-\alpha-\alpha'}{k}-2p-1}\frac{\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon+2pk)} \quad (41)$$

By using equations (17) and equation (10) in (41), we get

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)$$

$$=\sum_{p=0}^{\infty}\frac{x^{\frac{\sigma+\gamma-\alpha-\alpha'}{k}-1}(-c)^p(\frac{x}{2})^{\frac{\upsilon}{k}-2p}\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk)\Gamma_k(k-\sigma+\upsilon-\beta+2pk)}{\Gamma_k(pk+\upsilon+k)\Gamma_k(k-\sigma+\upsilon+\alpha-\beta+2pk)\Gamma_k(k-\sigma+\upsilon-\gamma+\alpha+\beta'+2pk+\alpha'k)\Gamma_k(k-\sigma+\upsilon+2pk)}. \tag{42}$$

By using equation (17) in equation (42), we get the final result

$$(I_{k,0^-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)=x^{\frac{1}{k}(\sigma+\upsilon+\gamma-\alpha-\alpha')-1}(2k)^{\frac{-\upsilon}{k}}$$

$${}_3\psi_4^k\left[\begin{array}{c}(k-\sigma+\upsilon-\beta,2),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k,2),(k-\sigma+\upsilon+\alpha+\beta'-\gamma,2)\\(\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon-\gamma+\alpha+\alpha'k+\beta',2),(k-\sigma+\upsilon-\gamma+\alpha-\beta,2),(k-\sigma+\upsilon,2)\end{array}\middle|-\frac{c}{4kx^2}\right].$$

□

Corollary 3.2. *Taking* $k=1, c=1$ *in Theorem* (3.1), *we get*

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}[t^{\sigma-1}J_{\upsilon}(\frac{1}{t})])(x))=x^{\sigma+\upsilon+\gamma-\alpha-\alpha'-1}2^{-\upsilon}$$

$$\times{}_3\psi_4\left[\begin{array}{c}(1-\sigma+\upsilon-\beta,2),(1-\sigma+\upsilon-\gamma+\alpha+\alpha',2),(1-\sigma+\upsilon+\alpha+\beta'-\gamma,2)\\(\upsilon+1,1),(1-\sigma+\upsilon-\gamma+\alpha+\alpha'+\beta',2),(1-\sigma+\upsilon-\gamma+\alpha-\beta,2),(1-\sigma+\upsilon,2)\end{array}\middle|\frac{-1}{4x^2}\right].$$

Theorem 3.3. *Let* $\alpha,\ \beta,\ \gamma,\ \upsilon,\ \sigma\in\mathbb{C},\ k>0,\ c\in\Re$ *and* $x>0$ *be such that* $\Re(\upsilon)>-1, \Re(\alpha)>0$ *and* $\Re(\frac{\sigma-\upsilon}{k})<1+min[\Re(\beta),\Re(\gamma)]$. *Then there holds the following relation:*

$$(I_{k,0^-}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x)=x^{\frac{1}{k}(\sigma-\upsilon-\beta)-1}(2k)^{\frac{-\upsilon}{k}}{}_2\psi_3^k\left[\begin{array}{c}(k-\sigma+\upsilon+\beta,2),(k-\sigma+\upsilon+\gamma,2)\\(\frac{\upsilon}{k}+1,1),(k-\sigma+\upsilon,2),(1-\sigma+\upsilon+\alpha+\beta+\gamma,2)\end{array}\middle|-\frac{c}{4kx^2}\right].$$

Proof. Consider the right sided Saigo fraction k-integral operator (4) with the product of power function with Bessel k-function (16), we have

$$\begin{aligned}(I_{k,0^-}^{\alpha,\beta,\gamma}[t^{\frac{\sigma}{k}-1}W_{\upsilon,c}^{k}(\frac{1}{t})])(x) &= \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{\Gamma_k(\alpha)}\frac{1}{k}\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}t^{\frac{-\alpha-\beta}{k}}{}_2F_{1,k}(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t})t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt\\ &= \mathcal{E}^{n,k}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_x^{\infty}t^{\frac{\alpha}{k}-1}(1-\frac{x}{t})^{\frac{\alpha}{k}-1}t^{\frac{-\alpha-\beta}{k}}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}(1-\frac{x}{t})^m t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt\\ &= \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_x^{\infty}(1-\frac{x}{t})^{\frac{\alpha}{k}+m-1}t^{\frac{\sigma-\upsilon+\alpha-\alpha-\beta}{k}-2n-2}\Big]dt.\end{aligned} \tag{43}$$

By putting $u=\frac{x}{t}\Rightarrow dt=-xu^{-2}du$ if $t=x\Rightarrow u=1$ if $t=\infty\Rightarrow u=0$ in (43), we obtain

$$(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_1^0(1-u)^{\frac{\alpha}{k}+m-1}(xu^{-1})^{\frac{\sigma-v-\beta}{k}-2n-2}\Big](-xu^{-2})du$$

$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-v-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{1}{k}\int_0^1 u^{\frac{k-\sigma+v+\beta+2nk}{k}-1}(1-u)^{\frac{\alpha+mk}{k}-1}\Big]du. \quad (44)$$

By using equation (7) in equation (44), we obtain

$$(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-v-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\beta_k(k-\sigma+v+\beta+2nk,\alpha+mk)]$$

$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-v-\beta}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}}{(\alpha)_{m,k}m!}\frac{\Gamma_k(k-\sigma+v+\beta+2nk)\Gamma_k(\alpha+mk)}{\Gamma_k(k-\sigma+v+\beta+2nk+\alpha+mk)}\Big]. \quad (45)$$

By using equations (12) and equation (14) in equation (45), we have

$$(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}x^{\frac{\sigma-v-\beta}{k}-2n-1}\sum_{m=0}^{\infty}\frac{(\alpha+\beta)_{m,k}(-\gamma)_{m,k}(1)^m}{(k-\sigma+v+\beta+2nk+\alpha)_{m,k}m!}\frac{\Gamma_k(k-\sigma+v+\beta+2nk)}{\Gamma_k(k-\sigma+v+\beta+2nk+\alpha)}$$

$$= \frac{\Gamma_k(k-\sigma+v+\gamma+2nk)\Gamma_k(k-\sigma+v+\beta+2nk)}{\Gamma_k(k-\sigma+v+2nk)\Gamma_k(k-\sigma+v+\alpha+\beta+2nk+\gamma)}. \quad (46)$$

By using equations (10) and equation (17) in (46), we get

$$(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x)$$

$$= x^{\frac{\sigma-v-\beta}{k}-2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{v}{k}}}{k^{\frac{v}{k}+1+2n-1}\Gamma(\frac{v}{k}+1+n)n!}\frac{\Gamma_k(k-\sigma+v+\gamma+2nk)\Gamma_k(k-\sigma+v+\beta+2nk)}{\Gamma_k(k-\sigma+v+2nk)\Gamma_k(k-\sigma+v+\alpha+\beta+\gamma+2nk)}$$

$$= \frac{x^{\frac{\sigma-v-\beta}{k}-1}}{(2k)^{\frac{v}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(k-\sigma+v+\gamma+2nk)}{\Gamma(\frac{v}{k}+1+n)\Gamma_k(k-\frac{\sigma}{+}v+2nk)}\frac{\Gamma_k(k-\sigma+v+\beta+2nk)}{\Gamma_k(k-\sigma+v+\alpha+\beta+\gamma+2nk)}\Big]\frac{(\frac{-c}{4kx^2})^n}{n!}. \quad (47)$$

By using equation (15) in equation (47), we get the final result

$$(I^{\alpha,\beta,\gamma}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-v-\beta)-1}(2k)^{\frac{-v}{k}}\,{}_2\psi^k_3\left[\begin{array}{c}(k-\sigma+v+\beta,2),(k-\sigma+v+\gamma,2)\\(\frac{v}{k}+1,1),(k-\sigma+v,2),(1-\sigma+v+\alpha+\beta+\gamma,2)\end{array}\Big|-\frac{c}{4kx^2}\right].$$

□

Theorem 3.4. *Let* $\alpha, v, \sigma \in \mathbb{C}$, $k > 0$, $c \in \Re$ *and* $x > 0$ *be such that* $\Re(v) > -1, \Re(\alpha) > 0$, *then there holds the following relation:*

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{v,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-v+\alpha)-1}(2k)^{\frac{-v}{k}}\,{}_1\psi^k_2\left[\begin{array}{c}(k-\sigma+v-\alpha,2)\\(\frac{v}{k}+1,1),(k-\sigma-v,2)\end{array}\Big|-\frac{c}{4kx^2}\right].$$

Proof. Consider the right sided Rieman Liuville fractional k-integral operator (6) with the product of power function and Bessel k-function (16), we have

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = \sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n+\frac{\upsilon}{k}}}{\Gamma_k(nk+\upsilon+k)n!}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}t^{\frac{\sigma-\upsilon}{k}-2n-1}\Big]dt$$
$$= \mathcal{E}^{n,k}\Big[\frac{1}{\Gamma_k(\alpha)}\frac{1}{k}\int_x^{\infty}(1-\frac{x}{t})^{\frac{\alpha}{k}-1}t^{\frac{\sigma-\upsilon+\alpha}{k}-2n-2}\Big]dt. \quad (48)$$

By putting $u=\frac{x}{t}\Rightarrow dt=-xu^{-2}du$ if $t=x\Rightarrow u=1$ and $t=\infty\Rightarrow u=0$ in (48), we have

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{1}{k\Gamma_k(\alpha)}\int_1^{0}(1-u)^{\frac{\alpha}{k}-1}(xu^{-1})^{\frac{\sigma-\upsilon+\alpha}{k}-2n-2}(-xu)^{-2}\Big]du$$
$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^{1}u^{\frac{k-\sigma+\upsilon-\alpha+2nk}{k}-1}(1-u)^{\frac{\alpha}{k}-1}\Big]du. \quad (49)$$

By using equation (7) in equation (49), we get

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\beta_k(k-\sigma+\upsilon-\alpha+2nk,\alpha)\Big].$$
$$= \mathcal{E}^{n,k}\Big[\frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)\Gamma_k(\alpha)}{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk+\alpha)}\Big]. \quad (50)$$

By using equations (10) and equation (17) in (50), we obtain

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = \frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{\Gamma_k(\alpha)}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{2n+\frac{\upsilon}{k}}\Gamma(n+\frac{\upsilon}{k}+1)n!}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)\Gamma_k(\alpha)}{\Gamma_k(k-\sigma+\upsilon+2nk)}$$
$$= x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}\sum_{n=0}^{\infty}\frac{(-c)^n(\frac{1}{2})^{2n}(\frac{1}{2})^{\frac{\upsilon}{k}}}{k^{2n+\frac{\upsilon}{k}}n!}\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)}{\Gamma(n+\frac{\upsilon}{k}+1)\Gamma_k(k-\sigma+\upsilon+2nk)}$$
$$= \frac{x^{\frac{\sigma-\upsilon+\alpha}{k}-2n-1}}{(2k)^{\frac{\upsilon}{k}}}\sum_{n=0}^{\infty}\Big[\frac{\Gamma_k(k-\sigma+\upsilon-\alpha+2nk)}{\Gamma(n+\frac{\upsilon}{k}+1)\Gamma_k(k-\sigma+\upsilon+2nk)}\Big]\frac{(\frac{-c}{4kx^2})^n}{n!}. \quad (51)$$

By using equation (15) in equation (51), we get the final result

$$(I^{\alpha}_{k,0^-}[t^{\frac{\sigma}{k}-1}W^k_{\upsilon,c}(\frac{1}{t})])(x) = x^{\frac{1}{k}(\sigma-\upsilon+\alpha)-1}(2k)^{\frac{-\upsilon}{k}}\,{}_1\psi^k_2\left[\begin{matrix}(k-\sigma+\upsilon-\alpha,2)\\(\frac{\upsilon}{k}+1,1),(k-\sigma-\upsilon,2)\end{matrix}\Big|-\frac{c}{4kx^2}\right].$$

□

Conclusion

In this paper, we have derived generalized k-fractional integral operators involving Appell k-function as its kernels with Bessel k-function. We have proved some composition formulae for Saigo, Riemann-Liouville k-fractional integral operators. The results have been established in terms of generalized k-Wright hypergeometric function. Furthermore if we take $k=1$, then we find out the results which are discussed in the form of corollaries (2.2) and (3.2).

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