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On Pythagorean Fuzzy Soft Boundary

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Abstract. The aim of this paper is to initiate the concept of pythagorean fuzzy soft (PFS) boundary. The characterizations and properties of PFS boundary are discussed and investigated in general as well as in terms of PFS interior and PFS closure.

1. Introduction

Many complicated ideas in the fields of economics, architecture, management , medical research, etc. require unknown data. These problems, which we face in our day-to-day lives, can not be solved by classical mathematical methods due to a large number of uncertainties. Decision-making is a vital activity for all those professions where professionals apply their expertise of a particular field to take effective decisions. However, owing to the various pressures of day-to-day life, decision-makers can not be able to offer their decisions in precisely crisp shape. Thus, in order to deal with it, they tend to use the fuzzy set theory [26] to offer their preferences under the ambiguous and imprecise existence. In this theory, the calculation of each element is achieved with the aid of the degree of membership. However, with increasing uncertainty, there is often a degree of hesitancy between the priorities of decision-making and thus the study performed in those conditions is not optimal. To fix this, the essential extension of the fuzzy set theory named as intuitionistic fuzzy set (IFS) proposed by Atanassov [1] inserted the degree of non-membership v in the analysis along with the degree of membership μ in such a way that $\mu + v \leq 1$. D. Coker [5] has developed and studied the concept of IF topological spaces and Hussain [8] studied intutionistic fuzzy soft boundary.

Intuitionistic fuzzy set theory is based on a limitation on decision-makers that they have assigned their desires only to the setting where the $\mu + v \leq 1$ limit is reached. However, if an expert gives 0.8 as membership and 0.3 as non-membership to an object, then it is obvious that $0.8 + 0.3 \nleq 1$ and therefore the above intuitionistic fuzzy set theory can not solve these problems. To overcome these problems, Yager [23, 24] introduced the concept of Pythagorean Fuzzy sets which is a generalization of intuitionistic fuzzy sets, by relaxing the condition $\mu + v \leq 1$ to $\mu^2 + v^2 \leq 1$. Thus, the pythagorean fuzzy sets treats far more information than the intuitionistic fuzzy sets. After that, some different studies are investigated using aggregation operators of pythagorean fuzzy sets.

In 1999, Molodtsov [13] introduced soft sets to address the lack of a parametrization tool when handling vagueness. Soft set theory is one of the most popular theories of recent times. Therefore, many researchers have made successful studies on soft set structure [16–19]. A soft set is a parameterized family of sets which

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has been extended into different hybrid structures such as fuzzy soft sets [11], intuitionistic fuzzy soft sets [12] and Pythagorean fuzzy soft sets [20]. Since the Pythagorean fuzzy set is extremely capable of dealing with vagueness or ambiguity, the parameterized Pythagorean fuzzy set family, which is Pythagorean fuzzy soft set, can also perform well. Recently, many studies on pythagorean fuzzy theory and pythagorean fuzzy soft theory have been conducted by researchers [2–4, 6, 7, 9, 14, 22]. Pythagorean fuzzy topological structure introduced by Olgun et al [15]. Also, Riaz et al. [21], Yolcu and Ozturk [25] studied Pythagorean fuzzy soft topological spaces.

In this paper, we initiate the concept of pythagorean fuzzy soft boundary. We discuss and explore the characterizations and properties of pythagorean fuzzy soft boundary in general as well as in terms of pythagorean fuzzy soft interior and pythagorean fuzzy soft closure. Examples and counterexamples are also presented to validate the discussed results.

2. Preliminaries

Definition 2.1. *[26] Let X be a universe. A fuzzy set F in X, $F = \{(x, \mu_F(x)) : x \in X\}$, where $\mu_F : X \to [0, 1]$ is the membership function of the fuzzy set F; $\mu_F(x) \in [0, 1]$ is the membership of $x \in X$ in f. The set of all fuzzy sets over X will be denoted by $FS(X)$.*

Definition 2.2. *[1] An intuitionistic fuzzy set F in X is $F = \{(x, \mu_F(x), v_F(x)) : x \in X\}$, where $\mu_F : X \to [0, 1]$, $v_F : X \to [0, 1]$ with the condition $0 \leq \mu_F(x) + v_F(x) \leq 1$, $\forall x \in X$. The numbers $\mu_F, v_F \in [0, 1]$ denote the degree of membership and non-membership of x to F, respectively. The set of all intuitionistic fuzzy sets over X will be denoted by $IFS(X)$.*

Definition 2.3. *[13] Let E be a set of parameters and X be the universal set. A pair (F, E) is called a soft set over X, where F is a mapping $F : E \to \mathcal{P}(X)$. In other words, the soft set is a parameterized family of subsets of the set X.*

Definition 2.4. *[11] Let E be a set of parameters and X be the universal set. A pair (F, E) is called a fuzzy soft set over X, If $F : E \to FS(X)$ is a mapping from E into the set of all fuzzy sets in X, where $FS(X)$ is the set of all fuzzy subset of X.*

Definition 2.5. *[12] Let X be an initial universe E be a set of parameters. A pair (F, E) is called an intuitionistic fuzzy soft set over X, where F is a mapping given by, $F : E \to IFS(X)$.*

In general, for every $e \in E$, $F(e)$ is an intuitionistic fuzzy set of X and it is called an intuitionistic fuzzy value set of parameter e. Clearly, $F(e)$ can be written as a intuitionistic fuzzy set such that $F(e) = \{(x, \mu_F(x), v_F(x)) : x \in X\}$

Definition 2.6. *[23] Let X be a universe of discourse. A pythagorean fuzzy set (PFS) in X is given by, $P = \{(x, \mu_P(x), v_P(x)) : x \in X\}$ where, $\mu_P : X \to [0, 1]$ denotes the degree of membership and $v_p : X \to [0, 1]$ denotes the degree of nonmembership of the element $x \in X$ to the set P with the condition that $0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1$.*

Definition 2.7. *[20] Let X be the universal set and E be a set of parameters. The pythagorean fuzzy soft set is defined as the pair (F, E) where, $F : E \to PFS(X)$ and $PFS(X)$ is the set of all Pythagorean fuzzy subsets of X. If $\mu_F^2(x) + v_F^2(x) \leq 1$ and $\mu_F(x) + v_F(x) \leq 1$, then pythagorean fuzzy soft sets degenerate into intuitionistic fuzzy soft sets.*

Definition 2.8. *[20] Let $A, B \subseteq E$ and (F, A), (G, B) be two pythagorean fuzzy soft sets over X. (F, A) is said to be pythagorean fuzzy soft subset of (G, B) denoted by $(F, A)\widetilde{\subseteq}(G, B)$ if,*

1. *$A \subseteq B$*
2. *$\forall e \in A$, $F(e)$ is a pythagorean fuzzy subset of $G(e)$ that is, $\forall x \in U$ and $\forall e \in A$, $\mu_{F(e)}(x) \leq \mu_{G(e)}(x)$ and $v_{F(e)}(x) \geq v_{G(e)}(x)$. If $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$ then (F, A), (G, B) are said to be equal.*

Definition 2.9. *[20] Let (F,E) two pythagorean fuzzy soft sets over X. The complement of (F,E) is denoted by $(F,E)^c$ and is defined by*

$$(F,E)^c = \{(e,(x,v_{F(e)}(x),\mu_{F(e)}(x)) : x \in X) : e \in E\}$$

Definition 2.10. *[10] a) A pythagorean fuzzy soft set (F,E) over the universe X is said to be a null pythagorean fuzzy soft set if $\mu_{F(e)}(x) = 0$ and $v_{F(e)}(x) = 1$; $\forall e \in E$, $\forall x \in X$. It is denoted by $\widetilde{0}_{(X,E)}$.*

b) A pythagorean fuzzy soft set (F,E) over the universe X is said to be an absolute pythagorean fuzzy soft set if $\mu_{F(e)}(x) = 1$ and $v_{F(e)}(x) = 0$; $\forall e \in E$, $\forall x \in X$. It is denoted by $\widetilde{1}_{(X,E)}$.

Definition 2.11. *[10] Let (F,A) and (G,B) be two pythagorean fuzzy soft sets over the universe set X, E be a parameter set and $A,B \subseteq E$.Then,*

a) Extended union of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cup}_E(G,B) = (H,C)$ where $C = A \cup B$ and (H,C) defined by;

$$(H,C) = \{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), \text{ if } e \in A - B \\ \mu_{G(e)}(x), \text{ if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), \text{ if } e \in A - B \\ v_{G(e)}(x), \text{ if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

b) Extended intersection of (F,A) and (G,B) is denoted by $(F,E)\widetilde{\cap}_E(G,B) = (H,C)$ where $C = A \cup B$ and (H,C) defined by;

$$(H,C) = \{(e,(x,\mu_{H(e)}(x),v_{H(e)}(x)) : x \in X) : e \in E\}$$

where

$$\mu_{H(e)}(x) = \begin{cases} \mu_{F(e)}(x), \text{ if } e \in A - B \\ \mu_{G(e)}(x), \text{ if } e \in B - A \\ \min\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

$$v_{H(e)}(x) = \begin{cases} v_{F(e)}(x), \text{ if } e \in A - B \\ v_{G(e)}(x), \text{ if } e \in B - A \\ \max\{\mu_{F(e)}(x), \mu_{G(e)}(x)\}, \text{ if } e \in A \cap B \end{cases}$$

Let X be an initial universe and $PFS(X)$ denote the family of pythagorean fuzzy sets over X and $PFSS(X,E)$ the family of all pythagorean fuzzy soft sets over X with parameters in E.

Definition 2.12. *[25]Let $X \neq \emptyset$ be a universe set and $\widetilde{\tau} \subset PFSS(X,E)$ be a collection of pythagorean fuzzy soft sets over X, then τ is said to be on pythagorean fuzzy soft topology on X if*

(i) $\widetilde{0}_{(X,E)}$, $\widetilde{1}_{(X,E)}$ belong to $\widetilde{\tau}$,

(ii) The union of any number of pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,

(iii) The intersection of any two pythagorean fuzzy soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triple $(X,\widetilde{\tau},E)_p$ is called an pythagorean fuzzy soft topological space over X. Every member of τ is called a pythagorean fuzzy soft open set in X.

Definition 2.13. *[25] Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X. A pythagorean fuzzy soft set (F,E) over X is said to be a pythagorean fuzzy soft closed set in X, if its complement $(F,E)^c$ belongs to $\widetilde{\tau}$.*

Definition 2.14. *[25]Let* $(X, \widetilde{\tau}, E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* (F, E) *be a pythagorean fuzzy soft sets over* X. *The pythagorean fuzzy soft closure of* (F, E) *denoted by* $pcl(F, E)$ *is the intersection of all pythagorean fuzzy soft closed super sets of* (F, E).

Clearly $pcl(F, E)$ *is the smallest pythagorean fuzzy soft closed set over* X *which contain* (F, E).

Theorem 2.15. *[25]Let* $(X, \widetilde{\tau}, E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(F, E) \in PFSS(X, E)$. *Then the following propeties hold.*

(i) $pcl(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ and $pcl(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $(F, E) \widetilde{\subseteq} pcl\,(F, E)$,
(iii) (F, E) is a pythagorean fuzzy soft closed set $\Leftrightarrow pcl(F, E) = (F, E)$,
(iv) $pcl\,(pcl\,(F, E)) = pcl(F, E)$,
(v) $(F, E) \widetilde{\subseteq} (G, E) \Rightarrow pcl\,(F, E) \widetilde{\subseteq} pcl\,(G, E)$,
(vi) $pcl\left((F, E) \widetilde{\cup}_E (G, E)\right) = pcl\,(F, E) \widetilde{\cup}_E pcl\,(G, E)$.

Definition 2.16. *[25]Let* $(X, \widetilde{\tau}, E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(H, E) \in PFSS(X, E)$. *The pythagorean fuzzy soft interior of* (H, E), *denoted by* $pint(H, E)$, *is the union of all the pythagorean fuzzy soft open sets contained in* (H, E).

Theorem 2.17. *[25]Let* $(X, \widetilde{\tau}, E)_p$ *be a pythagorean fuzzy soft topological space over* X *and* $(H, E) \in PFSS(X, E)$. *Then the following properties hold.*
(i) $pint(\widetilde{0}_{(X,E)}) = \widetilde{0}_{(X,E)}$ *and* $pint(\widetilde{1}_{(X,E)}) = \widetilde{1}_{(X,E)}$,
(ii) $pint\,(H, E) \widetilde{\subseteq} (H, E)$,
(iii) (H, E) *is a pythagorean fuzzy soft open set* $\Leftrightarrow pint(H, E) = (H, E)$,
(iv) $pint\,(pint\,(H, E)) = pint(H, E)$,
(v) $(H, E) \widetilde{\subseteq} (G, E) \Rightarrow pint\,(H, E) \widetilde{\subseteq} pint\,(G, E)$,
(vi) $pint\left((H, E) \widetilde{\cap}_E (G, E)\right) = pint\,(H, E) \widetilde{\cap}_E pint\,(G, E)$.

3. Pythagorean Fuzzy Soft Boundary

Definition 3.1. *The difference of two pythagorean fuzzy soft sets* (F, E) *and* (G, E) *over* X, *denoted by* $(F, E)\widetilde{\setminus}(G, E)$ *and defined by* $(F, E)\widetilde{\setminus}(G, E) = (F, E)\widetilde{\cap}_E(G, E)^c$

Example 3.2. *Let* (F, E) *and* (G, E) *be two pythagorean fuzzy soft set defined as follows;*

$$(F, E) = \left\{ \begin{array}{c} (e_1, \{(x_1, 0.3, 0.5), (x_2, 0.2, 0.6)\}) \\ (e_2, \{(x_1, 0.4, 0.1), (x_2, 0.5, 0.6)\}) \end{array} \right\}$$

$$\begin{aligned} (G, E) &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.4, 0.8), (x_2, 0.9, 0.2)\}) \\ (e_2, \{(x_1, 0.6, 0.3), (x_2, 0.7, 0.4)\}) \end{array} \right\} \\ (G, E)^c &= \left\{ \begin{array}{c} (e_1, \{(x_1, 0.8, 0.4), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.3, 0.6), (x_2, 0.4, 0.7)\}) \end{array} \right\} \end{aligned}$$

Then $(F, E)\widetilde{\setminus}(G, E) = (F, E)\widetilde{\cap}_E(G, E)^c$ *and we find*

$$(F, E)\widetilde{\cap}_E(G, E)^c = \left\{ \begin{array}{c} (e_1, \{(x_1, 0.3, 0.5), (x_2, 0.2, 0.9)\}) \\ (e_2, \{(x_1, 0.3, 0.6), (x_2, 0.4, 0.7)\}) \end{array} \right\}.$$

Definition 3.3. *[21] Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then the pythagorean fuzzy soft boundary of (F,E), denoted by $Fr(F,E)$ and defined as $Fr(F,E) = pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right)$.*

Example 3.4. *Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2\}$ and*

$$\widetilde{\tau} = \left\{\widetilde{0}_{(X,E)}, \widetilde{1}_{(X,E)}, (F_1,E), (F_2,E)\right\}$$

where $(F_1,E), (F_2,E)$ are pythagorean fuzzy soft sets over X, defined as;

$$\begin{aligned}(F_1,E) &= \left\{\begin{array}{l}(e_1,\{(x_1,0.6,0.2),(x_2,0.8,0.4)\})\\(e_2,\{(x_1,0.7,0.3),(x_2,0.5,0.2)\})\end{array}\right\}\\(F_2,E) &= \left\{\begin{array}{l}(e_1,\{(x_1,0.7,0.2),(x_2,0.9,0.2)\})\\(e_2,\{(x_1,0.8,0.2),(x_2,0.7,0.1)\})\end{array}\right\}\end{aligned}$$

Then $(X,\widetilde{\tau},E)_p$ is a pythagorean fuzzy soft topological spaces on X.The members of $\widetilde{\tau}$ obviously pythagorean fuzzy open sets. Now, we find pythagorean fuzzy closed sets;

$$\begin{aligned}\widetilde{0}^c_{(X,E)} &= \widetilde{1}_{(X,E)}\\\widetilde{1}^c_{(X,E)} &= \widetilde{0}_{(X,E)}\end{aligned}$$

$$(F_1,E)^c = \left\{\begin{array}{l}(e_1,\{(x_1,0.2,0.6),(x_2,0.4,0.8)\})\\(e_2,\{(x_1,0.3,0.7),(x_2,0.2,0.5)\})\end{array}\right\}$$

$$(F_2,E)^c = \left\{\begin{array}{l}(e_1,\{(x_1,0.2,0.7),(x_2,0.2,0.9)\})\\(e_2,\{(x_1,0.2,0.8),(x_2,0.1,0.7)\})\end{array}\right\}$$

We consider the pythagorean fuzzy soft set $(G,E)\widetilde{\subset}PFSS(X,E)$.

$$(G,E) = \left\{\begin{array}{l}(e_1,\{(x_1,0.1,0.8),(x_2,0.2,0.9)\})\\(e_2,\{(x_1,0.1,0.9),(x_2,0.1,0.7)\})\end{array}\right\}$$

so that

$$(G,E)^c = \left\{\begin{array}{l}(e_1,\{(x_1,0.8,0.1),(x_2,0.9,0.2)\})\\(e_2,\{(x_1,0.9,0.1),(x_2,0.7,0.1)\})\end{array}\right\}$$

Obviously, $\widetilde{0}^c_{(X,E)}, \widetilde{1}^c_{(X,E)}, (F_1,E)^c, (F_2,E)^c$ are all pythagorean fuzzy soft closed sets over $(X,\widetilde{\tau},E)_p$. Then

$$\widetilde{1}^c_{(X,E)}, (F_1,E)^c, (F_2,E)^c\widetilde{\supset}(G,E).$$

Therefore $pcl(F,E) = \widetilde{1}^c_{(X,E)}\widetilde{\cap}_E(F_1,E)^c\widetilde{\cap}_E(F_2,E)^c = (F_2,E)^c$. Also we find $pcl\left((F,E)^c\right) = \widetilde{1}_{(X,E)}$.
So, $Fr(F,E) = pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right) = (F_2,E)^c\widetilde{\cap}_E\widetilde{1}_{(X,E)} = (F_2,E)^c$, Hence

$$Fr(F,E) = \left\{\begin{array}{l}(e_1,\{(x_1,0.2,0.7),(x_2,0.2,0.9)\})\\(e_2,\{(x_1,0.2,0.8),(x_2,0.1,0.7)\})\end{array}\right\}.$$

Theorem 3.5. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then the following properties hold;*

1. $(Fr(F,E))^c = pint(F,E)\widetilde{\cup}_E pint\left((F,E)^c\right)$
2. $pcl(F,E) = pint(F,E)\widetilde{\cup}_E Fr(F,E)$

3. $Fr(F,E) = pcl(F,E)\tilde{\setminus} pint(F,E)$
4. $pint(F,E) = (F,E)\tilde{\setminus} Fr(F,E)$
5. $Fr(pcl(F,E)) \tilde{\subseteq} Fr(F,E)$
6. $Fr(F,E)\widetilde{\cap}_E pint(F,E) = \widetilde{0}_{(X,E)}$.
7. $pcl(pint(F,E)) = (F,E)\tilde{\setminus} Fr(F,E)$

Proof. *(1)*

$$\begin{aligned} pint(F,E)\widetilde{\cup}_E pint((F,E)^c) &= \left((pint(F,E))^c\right)^c \widetilde{\cup}_E \left((pint((F,E)^c))^c\right)^c \\ &= \left[(pint(F,E))^c \widetilde{\cap}_E (pint((F,E)^c))^c\right]^c \\ &= \left[pcl((F,E)^c) \widetilde{\cap}_E pcl(F,E)\right]^c \\ &= (Fr(F,E))^c . \end{aligned}$$

(2)

$$\begin{aligned} pint(F,E)\widetilde{\cup}_E Fr(F,E) &= pint(F,E)\widetilde{\cup}_E \left(pcl(F,E)\widetilde{\cap}_E pcl((F,E)^c)\right) \\ &= \left[pint(F,E)\widetilde{\cup}_E pcl(F,E)\right] \widetilde{\cap}_E \left[pint(F,E)\widetilde{\cup}_E pcl((F,E)^c)\right] \\ &= pcl(F,E)\widetilde{\cap}_E \left[pint(F,E)\widetilde{\cup}_E (pint(F,E))^c\right] \\ &= pcl(F,E)\widetilde{\cap}_E \left(pint(F,E)\widetilde{\cup}_E (pint(F,E))^c\right) \\ &= pcl(F,E)\widetilde{\cap}_E \widetilde{1}_{(X,E)} \\ &= pcl(F,E). \end{aligned}$$

(3)

$$\begin{aligned} Fr(F,E) &= pcl(F,E)\widetilde{\cap}_E pcl((F,E)^c) \\ &= pcl(F,E)\widetilde{\cap}_E (pint(F,E))^c \\ &= pcl(F,E)\tilde{\setminus} pint(F,E). \end{aligned}$$

(4)

$$\begin{aligned} (F,E)\tilde{\setminus} Fr(F,E) &= (F,E)\widetilde{\cap}_E Fr((F,E)^c) \\ &= (F,E)\widetilde{\cap}_E \left(pint(F,E)\widetilde{\cup}_E pint((F,E)^c)\right) (by\ (1)) \\ &= \left[(F,E)\widetilde{\cap}_E pint(F,E)\right] \widetilde{\cup}_E \left[(F,E)\widetilde{\cap}_E pint((F,E)^c)\right] \\ &= pint(F,E)\widetilde{\cup}_E \widetilde{0}_{(X,E)}. \\ &= pint(F,E). \end{aligned}$$

(5)

$$\begin{aligned} Fr(pcl(F,E)) &= pcl(pcl(F,E)) \tilde{\setminus} pint(pcl(F,E)) \\ &= pcl(F,E)\tilde{\setminus} pint(pcl(F,E)) \\ &\tilde{\subseteq} pcl(F,E)\tilde{\setminus} pint(F,E) \\ &= Fr(F,E). \end{aligned}$$

(6) is similar to (3)
(7) can be easily obtained from the definition of a pythagorean fuzzy soft boundary. □

Remark 3.6. *By (3) of above Theorem 3.5, it is clear that* $Fr(F,E) = Fr((F,E)^c)$.

Theorem 3.7. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\sqsubset}PFSS(X,E)$. Then;*

1. *(F,E) is a pythagorean fuzzy open set over X if and only if $(F,E)\widetilde{\sqcap}_E Fr(F,E)=\widetilde{0}_{(X,E)}$,*
2. *(F,E) is a pythagorean fuzzy closed set over X if and only if $Fr(F,E)\widetilde{\sqsubset}(F,E)$,*
3. *If (G,E) be a pythagorean fuzzy closed (respt. open) set of an pythagorean fuzzy soft topological space with $(F,E)\widetilde{\sqsubset}(G,E)$, then $Fr(F,E)\widetilde{\sqsubset}(G,E)$ (respt. $Fr(F,E)\widetilde{\sqsubset}(G,E)^c$).*

Proof. *(1) Let (F,E) be an pythagorean fuzzy soft open set over X. Then $pint(F,E)=(F,E)$ implies that $(F,E)\widetilde{\sqcap}_E Fr(F,E)=pint(F,E)\widetilde{\sqcap}_E Fr(F,E)=\widetilde{0}_{(X,E)}$.*

Conversely, let $(F,E)\widetilde{\sqcap}_E Fr(F,E)=\widetilde{0}_{(X,E)}$. Then $(F,E)\widetilde{\sqcap}_E pcl(F,E)\widetilde{\sqcap}_E pcl((F,E)^c)=\widetilde{0}_{(X,E)}$ or $(F,E)\widetilde{\sqcap}_E pcl((F,E)^c)=\widetilde{0}_{(X,E)}$ or $pcl((F,E)^c)\widetilde{\sqsubset}(F,E)^c$, which implies $(F,E)^c$ is a pythagorean fuzzy closed and hence (F,E) is pythagorean fuzzy open set.

(2) Let (F,E) be an pythagorean fuzzy soft closed set over X. Then $pcl(F,E)=(F,E)$. Now $Fr(F,E)=pcl(F,E)\widetilde{\sqcap}_E pcl((F,E)^c)\widetilde{\sqsubset}pcl(F,E)=(F,E)$. That is, $Fr(F,E)\widetilde{\sqsubset}(F,E)$.

Conversely, $Fr(F,E)\widetilde{\sqsubset}(F,E)$. Then $Fr(F,E)\widetilde{\sqcap}_E(F,E)^c=\widetilde{0}_{(X,E)}$. Since $Fr(F,E)=Fr((F,E)^c)=\widetilde{0}_{(X,E)}$, we have $Fr((F,E)^c)\widetilde{\sqcap}_E(F,E)^c=\widetilde{0}_{(X,E)}$. By (1), $(F,E)^c$ is pythagorean fuzzy open set and hence (F,E) is pythagorean fuzzy closed set.

(3) $(F,E)\widetilde{\sqsubset}(G,E)$ follows that $pcl(F,E)\widetilde{\sqsubset}pcl(G,E)$. Since (G,E) is pythagorean fuzzy soft closed, then we get $Fr(F,E)=pcl(F,E)\widetilde{\sqcap}_E pcl((F,E)^c)\widetilde{\sqsubset}pcl(G,E)\widetilde{\sqcap}_E pcl((F,E)^c)\widetilde{\sqsubset}pcl(G,E)=(G,E)$. Similarly for the other inclusion. □

Theorem 3.8. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\widetilde{\sqsubset}PFSS(X,E)$. Then the following properties hold.*

1. $Fr\Big((F,E)\widetilde{\sqcup}_E(G,E)\Big)\widetilde{\sqsubset}Fr\Big((F,E)\widetilde{\sqcap}_E((G,E)^c)\Big)\widetilde{\sqcup}_E\Big[Fr(G,E)\widetilde{\sqcap}_E pcl((F,E)^c)\Big]$,
2. $Fr\Big((F,E)\widetilde{\sqcap}_E(G,E)\Big)\widetilde{\sqsubset}Fr\Big((F,E)\widetilde{\sqcap}_E pcl(G,E)\Big)\widetilde{\sqcup}_E\Big[Fr(G,E)\widetilde{\sqcap}_E pcl(F,E)\Big]$.

Proof. *(1)*

$$\begin{aligned}
Fr\Big((F,E)\widetilde{\sqcup}_E(G,E)\Big) &= pcl\Big((F,E)\widetilde{\sqcup}_E(G,E)\Big)\widetilde{\sqcap}_E pcl\Big(\big((F,E)\widetilde{\sqcup}_E(G,E)\big)^c\Big)\\
&= \Big(pcl(F,E)\widetilde{\sqcup}_E pcl(G,E)\Big)\widetilde{\sqcap}_E pcl\Big((F,E)^c\widetilde{\sqcap}_E(G,E)^c\Big)\\
&\widetilde{\sqsubset} \Big(pcl(F,E)\widetilde{\sqcup}_E pcl(G,E)\Big)\widetilde{\sqcap}_E\Big[pcl((F,E)^c)\widetilde{\sqcap}_E pcl((G,E)^c)\Big]\\
&= \Big(pcl(F,E)\widetilde{\sqcup}_E pcl(G,E)\Big)\widetilde{\sqcap}_E pcl((G,E)^c)\widetilde{\sqcup}_E pcl(G,E)\\
&\quad \widetilde{\sqcap}_E\Big[pcl((F,E)^c)\widetilde{\sqcap}_E pcl((G,E)^c)\Big]\\
&= \Big(Fr(F,E)\widetilde{\sqcap}_E pcl((G,E)^c)\Big)\widetilde{\sqcup}_E\Big(Fr(G,E)\widetilde{\sqcap}_E pcl((F,E)^c)\Big)\\
&\widetilde{\sqsubset} Fr(F,E)\widetilde{\sqcup}_E Fr(G,E).
\end{aligned}$$

(2)

$$\begin{aligned}
Fr\Big((F,E)\widetilde{\sqcap}_E(G,E)\Big) &= pcl\Big((F,E)\widetilde{\sqcap}_E(G,E)\Big)\widetilde{\sqcap}_E pcl\Big(\big((F,E)\widetilde{\sqcap}_E(G,E)\big)^c\Big)\\
&\quad \widetilde{\sqsubset}\Big(pcl(F,E)\widetilde{\sqcup}_E pcl(G,E)\Big)\widetilde{\sqcap}_E pcl\Big((F,E)^c\widetilde{\sqcup}_E(G,E)^c\Big)\\
&= \Big(pcl(F,E)\widetilde{\sqcup}_E pcl(G,E)\Big)\widetilde{\sqcap}_E\Big[pcl((F,E)^c)\widetilde{\sqcup}_E pcl((G,E)^c)\Big]\\
&= \Big[\Big(pcl(F,E)\widetilde{\sqcap}_E pcl(G,E)\Big)\widetilde{\sqcap}_E pcl((G,E)^c)\Big]\widetilde{\sqcup}_E[(pcl(F,E)\\
&\quad \widetilde{\sqcap}_E pcl(G,E))\widetilde{\sqcap}_E pcl((G,E)^c)]\\
&= \Big(Fr(F,E)\widetilde{\sqcap}_E Fr(G,E)\Big)\widetilde{\sqcup}_E\Big(pcl(F,E)\widetilde{\sqcap}_E Fr((G,E))\Big)
\end{aligned}$$

□

Corollary 3.9. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\widetilde{\subset}PFSS(X,E)$. Then $Fr\left((F,E)\widetilde{\cap}_E(G,E)\right)\widetilde{\subset}Fr(F,E)\widetilde{\cap}_E Fr(G,E)$.*

Theorem 3.10. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E)\widetilde{\subset}PFSS(X,E)$. Then we have $Fr\,(Fr\,(Fr(F,E))) = Fr\,(Fr(F,E))$.*

Proof.

$$\begin{aligned} Fr\,(Fr\,(Fr(F,E))) &= pcl\,(Fr\,(Fr(F,E)))\,\widetilde{\cap}_E pcl\,((Fr\,(Fr(F,E)))^c) \\ &= (Fr\,(Fr(F,E)))\,\widetilde{\cap}_E pcl\,((Fr\,(Fr(F,E)))^c) \end{aligned}$$

Now consider

$$\begin{aligned} (Fr\,((Fr(F,E)))^c) &= \left[pcl\,(Fr(F,E))\,\widetilde{\cap}_E\,(Fr(F,E))^c\right]^c \\ &= \left[Fr(F,E)\widetilde{\cap}_E pcl\,((Fr(F,E))^c)\right]^c \\ &= (Fr(F,E))^c\,\widetilde{\cup}_E\,(pcl\,((Fr(F,E))^c))^c \end{aligned}$$

Therefore

$$\begin{aligned} pcl\,((Fr\,(Fr(F,E)))^c) &= pcl\left(\left[pcl\,((Fr(F,E))^c)\,\widetilde{\cup}_E\,(pcl\,((Fr(F,E))^c))^c\right]\right) \\ &= pcl\,(pcl\,((Fr(F,E))^c))\,\widetilde{\cup}_E pcl\left((pcl\,((Fr(F,E))^c))^c\right) \\ &= (G,E)\widetilde{\cup}_E\left((pcl\,((Fr(G,E))^c))^c\right) = \widetilde{1}_{(X,E)} \end{aligned}$$

where $(G,E) = pcl\,((pcl\,((Fr(F,E))^c)))$. From the above equations, we have

$$Fr\,(Fr\,(Fr(F,E))) = Fr\,(Fr(F,E))\,\widetilde{\cap}_E\widetilde{1}_{(X,E)} = Fr\,(Fr(F,E))$$

□

Theorem 3.11. *Let $(X,\widetilde{\tau},E)_p$ be a pythagorean fuzzy soft topological space over X and $(F,E),(G,E)\widetilde{\subset}PFSS(X,E)$. Then the following properties hold.*

1. $\left((F,E)\widetilde{\setminus}pint(G,E)\right)\widetilde{\subset}pint(F,E)\widetilde{\setminus}pint(G,E)$
2. $Fr\,(pint(F,E))\,\widetilde{\subset}Fr(F,E)$.

Proof. (1)

$$\begin{aligned} \left((F,E)\widetilde{\setminus}pint(G,E)\right) &= \left((F,E)\widetilde{\cap}_E pint\,((G,E)^c)\right) \\ &= pint(F,E)\widetilde{\cap}_E pint\,((G,E)^c) \\ &= pint(F,E)\widetilde{\cap}_E\,(pcl(G,E))^c \\ &= pint(F,E)\widetilde{\setminus}pcl(G,E) \\ &\widetilde{\subset}\; pint(F,E)\widetilde{\setminus}pint(G,E). \end{aligned}$$

(2)

$$\begin{aligned} Fr\,(pint(F,E)) &= pcl\,(pint(F,E))\,\widetilde{\cap}_E pcl\left((pint(F,E))^c\right) \\ &\widetilde{\subset}\; pcl\,(pint(F,E))\,\widetilde{\cap}_E pcl\,(pcl\,((F,E)^c)) \\ &\widetilde{\subset}\; pcl(F,E)\widetilde{\cap}_E pcl\,((F,E)^c) = Fr(F,E). \end{aligned}$$

□

Theorem 3.12. *Let $(X, \widetilde{\tau}, E)_p$ be a pythagorean fuzzy soft topological space over X and $(F, E)\widetilde{\subset}PFSS(X, E)$. Then $Fr(F, E) = \widetilde{0}_{(X,E)}$ if and only if (F, E) is both a pythagorean fuzzy soft closed and pythagorean fuzzy soft open set.*

Proof. *Suppose that $Fr(F, E) = \widetilde{0}_{(X,E)}$. Firstly, we show that (F, E) is a pythagorean fuzzy soft closed set.*

$$\begin{aligned} Fr(F,E) &= \widetilde{0}_{(X,E)} \Rightarrow pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right) = \widetilde{0}_{(X,E)} \\ &\Rightarrow pcl(F,E)\widetilde{\subset}\left(pcl\left((F,E)^c\right)\right)^c = pint(F,E) \\ &\Rightarrow pcl(F,E)\widetilde{\subset}(F,E) \Rightarrow pcl(F,E) = (F,E) \end{aligned}$$

This implies that (F, E) is pythagorean fuzzy soft closed set.

Now, we prove that (F, E) is a pythagorean fuzzy soft open set.

$$Fr(F,E) = \widetilde{0}_{(X,E)} \Rightarrow pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right)$$

or

$$\begin{aligned} (F,E)\widetilde{\cap}_E\left(pint(F,E)\right)^c &= \widetilde{0}_{(X,E)} \Rightarrow (F,E)\widetilde{\subset}pint(F,E) \\ &\Rightarrow pint(F,E) = (F,E) \end{aligned}$$

This implies that (F, E) is pythagorean fuzzy soft open set.

Conversely, suppose that (F, E) is both pythagorean fuzzy soft open and pythagorean fuzzy soft closed set. Then

$$\begin{aligned} Fr(F,E) &= pcl(F,E)\widetilde{\cap}_E pcl\left((F,E)^c\right) \\ &= pcl(F,E)\widetilde{\cap}_E\left(pint(F,E)\right)^c \\ &= (F,E)\widetilde{\cap}_E(F,E)^c = \widetilde{0}_{(X,E)}. \end{aligned}$$

□

4. Conclusion

In this paper, we introduced the concept of the pythagorean fuzzy soft boundary. We discussed and investigated the characteristics and properties of pythagorean fuzzy soft boundary in general as well as pythagorean fuzzy soft interior and pythagorean fuzzy soft closure. Examples and counterexamples are also given to verify the findings discussed. We will research more topological structures in pythagorean fuzzy soft sets in future studies. We hope that this study will be useful for the paper to be done in the theory of pythagorean fuzzy soft.

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