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New Integral Inequalities of Ostrowski Type for Quasi-Convex Functions with Applications

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Abstract. In this paper some new Ostrowski-type inequalities for functions whose derivatives in absolute values are quasi-convex are established. Some applications to special means of real numbers and applications for P.D.F.'s are given. We also give some applications of our results to get new error bounds for the sum of the midpoint formula.

1. Introduction

We recall that the notion of quasi-convex functions as following.

Definition 1.1. *(See [7]) A function* $f:[a,b]\to\mathbb{R}$ *is said to be quasi-convex on* $[a,b]$ *if*

$$f(tx+(1-t)y)\leq\max\{f(x),f(y)\},\ \text{for all } x,y\in[a,b].$$

It is to be noted that any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see e.g. [2]-[6]).

Let $f:I\subset[0,\infty]\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f'\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'(x)|\leq M$, then the following inequality holds (see [8]).

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]\tag{1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1), see [2] and the references therein.

In [4], Alomari and Darus proved several inequalities of Ostrowski type for quasi-convex functions, we will mention some them as following.

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Theorem 1.2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|$ is quasi-convex function on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \frac{(x-a)^2}{2(b-a)} \max\left\{ \left|f'(x)\right|, \left|f'(a)\right| \right\} + \frac{(b-x)^2}{2(b-a)} \max\left\{ \left|f'(x)\right|, \left|f'(b)\right| \right\}
\end{aligned}$$

for each $x \in [a,b]$.

Theorem 1.3. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\
&\leq \left(\frac{(b-x)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}} \left(\max\left\{ \left|f'(b)\right|^q, \left|f'(x)\right|^q \right\} \right)^{\frac{1}{q}} \\
&+ \left(\frac{(x-a)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}} \left(\max\left\{ \left|f'(a)\right|^q, \left|f'(x)\right|^q \right\} \right)^{\frac{1}{q}}
\end{aligned}$$

for each $x \in [a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

The main aim of this paper is to establish some new inequalities of Ostrowski type for quasi-convex functions and to give some deduced results to the celebrated Hadamard integral inequality. Based on these results, we obtain several applications for special means of real numbers, numerical integration and P.D.F.

2. Main Results

To prove our results we need the following Lemma:

Lemma 2.1. *(See [1]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° , $a,b \in I^\circ$ with $a<b$ and $f' \in L([a,b])$. Then*

$$\begin{aligned}
&f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \\
&= \frac{(x-a)^2}{4(b-a)} \left(\int_0^1 t f'\left(t\frac{a+x}{2} + (1-t)a \right) dt \right. \\
&\left. + \int_0^1 (1+t) f'\left(tx + (1-t)\frac{a+x}{2} \right) dt \right) \\
&- \frac{(b-x)^2}{4(b-a)} \left(\int_0^1 (2-t) f'\left(t\frac{b+x}{2} + (1-t)x \right) dt \right. \\
&\left. + \int_0^1 (1-t) f'\left(tb + (1-t)\frac{b+x}{2} \right) dt \right).
\end{aligned}$$

By using the Lemma 2.1 the following results can be obtained:

Theorem 2.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|$ is quasi-convex function on $[a,b]$, then one has the following inequality:*

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{(x-a)^2}{8(b-a)} \max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|, \left|f'(a)\right|\right\} \\ &+ \frac{3(x-a)^2}{8(b-a)} \max\left\{\left|f'(x)\right|, \left|f'\left(\frac{a+x}{2}\right)\right|\right\} \\ &+ \frac{3(b-x)^2}{8(b-a)} \max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|, \left|f'(x)\right|\right\} \\ &+ \frac{(b-x)^2}{8(b-a)} \max\left\{\left|f'(b)\right|, \left|f'\left(\frac{b+x}{2}\right)\right|\right\}, \end{aligned} \tag{2}$$

for all $x \in [a,b]$.

Proof. From the integral identity that is given in Lemma 2.1 and by using the properties of modulus, we can write

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{(x-a)^2}{4(b-a)} \left(\int_0^1 t\left|f'\left(t\frac{a+x}{2} + (1-t)a\right)\right| dt\right. \\ &+ \left.\int_0^1 (1+t)\left|f'\left(tx + (1-t)\frac{a+x}{2}\right)\right| dt\right) \\ &- \frac{(b-x)^2}{4(b-a)} \left(\int_0^1 (2-t)\left|f'\left(t\frac{b+x}{2} + (1-t)x\right)\right| dt\right. \\ &+ \left.\int_0^1 (1-t)\left|f'\left(tb + (1-t)\frac{b+x}{2}\right)\right| dt\right). \end{aligned} \tag{3}$$

By using quasi-convexity of $\left|f'\right|$, we have

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\ &\leq \frac{(x-a)^2}{4(b-a)} \max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|, \left|f'(a)\right|\right\} \int_0^1 t\,dt \\ &+ \frac{(x-a)^2}{4(b-a)} \max\left\{\left|f'(x)\right|, \left|f'\left(\frac{a+x}{2}\right)\right|\right\} \int_0^1 (1+t)\,dt \\ &+ \frac{(b-x)^2}{4(b-a)} \max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|, \left|f'(x)\right|\right\} \int_0^1 (2-t)\,dt \\ &+ \frac{(b-x)^2}{4(b-a)} \max\left\{\left|f'(b)\right|, \left|f'\left(\frac{b+x}{2}\right)\right|\right\} \int_0^1 (1-t)\,dt, \end{aligned} \tag{4}$$

for all $x \in [a,b]$.

By using the facts that

$$\int_0^1 (1+t)\,dt = \int_0^1 (2-t)\,dt = \frac{3}{2}$$
$$\int_0^1 t dt = \int_0^1 (1-t)\,dt = \frac{1}{2}$$

we get the inequality (2). This completes the proof of the theorem. □

An immediate consequence of Theorem 2.2 is the following:

Corollary 2.3. *If all the assumptions of Theorem 2.2 are satisfied and if we choose* $x = \frac{a+b}{2}$*, we get the following inequality:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \frac{b-a}{32}\left[\max\left\{ \left| f'\left(\frac{3a+b}{4}\right)\right|, \left|f'(a)\right| \right\} \right. \\ &+3\max\left\{ \left| f'\left(\frac{a+b}{2}\right)\right|, \left| f'\left(\frac{3a+b}{4}\right)\right| \right\} \\ &+3\max\left\{ \left| f'\left(\frac{a+3b}{4}\right)\right|, \left| f'\left(\frac{a+b}{2}\right)\right| \right\} \\ &\left. +\max\left\{ \left|f'(b)\right|, \left| f'\left(\frac{a+3b}{4}\right)\right| \right\} \right] \end{aligned} \tag{5}$$

Additionally,

1. *If* $|f'|$ *is increasing, then*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \frac{b-a}{32}\left[\left| f'\left(\frac{3a+b}{4}\right)\right| + 3\left| f'\left(\frac{a+b}{2}\right)\right| + 3\left| f'\left(\frac{a+3b}{4}\right)\right| + \left|f'(b)\right| \right]. \end{aligned} \tag{6}$$

2. *If* $|f'|$ *is decreasing, then*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ &\leq \frac{b-a}{32}\left[\left|f'(a)\right| + 3\left| f'\left(\frac{3a+b}{4}\right)\right| + 3\left| f'\left(\frac{a+b}{2}\right)\right| + \left| f'\left(\frac{a+3b}{4}\right)\right| \right]. \end{aligned} \tag{7}$$

Corollary 2.4. *If all the assumptions of Theorem 2.2 are satisfied and if we choose* $x = a$ *and* $x = b$*, respectively, we get the following inequalities:*

$$\begin{aligned} &\left| f(a) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq &\frac{3(b-a)}{8}\max\left\{ \left|f'(a)\right|, \left| f'\left(\frac{a+b}{2}\right)\right| \right\} \\ &+\frac{(b-a)}{8}\max\left\{ \left| f'\left(\frac{a+b}{2}\right)\right|, \left|f'(b)\right| \right\} \end{aligned}$$

and

$$\begin{aligned}&\left|f(b)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\\&\leq \frac{(b-a)}{8}\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|,\left|f'(a)\right|\right\}\\&+\frac{3(b-a)}{8}\max\left\{\left|f'(b)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}.\end{aligned}$$

Additionally, if we add these inequalities and by choosing $|f'|$ is increasing and decreasing, respectively, then we obtain:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-a)}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right]$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{(b-a)}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(a)\right|\right].$$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following theorem.

Theorem 2.5. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \qquad (8)\\&\leq\frac{1}{4(b-a)(p+1)^{\frac{1}{p}}}\left\{(x-a)^2\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\&+(x-a)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\&+(b-x)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}}\\&\left.+(b-x)^2\left(\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{b+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

for all $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and by using the Hölder integral inequality, we get

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \qquad (9)\\
&\le \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1+t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (2-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}
$$

for all $x\in[a,b]$.
Since $\left|f'\right|^q$ is quasi-convex on $[a,b]$, we know

$$\int_0^1\left|f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt\le\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}. \qquad (10)$$

Similarly,

$$\int_0^1\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\le\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}, \qquad (11)$$

$$\int_0^1\left|f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt\le\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\} \qquad (12)$$

and

$$\int_0^1\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\le\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{b+x}{2}\right)\right|^q\right\}. \qquad (13)$$

Using these inequalities in (9) and by making use of the necessary computations, the desired result is obtained. □

The following corollary is an immediate consequence of Theorem 2.5:

Corollary 2.6. *Suppose all the assumptions of Theorem 2.5 are satisfied. If we choose* $x=\frac{a+b}{2}$, *we get the following inequality:*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \qquad (14)\\
&\le\frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q,\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{a+3b}{4}\right)\right|^q,\left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Additionally,

1. *If* $|f'|^q$ *is increasing, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{15}$$
$$\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right\}.$$

2. *If* $|f'|^q$ *is decreasing, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{16}$$
$$\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^q+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right.$$
$$\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right].$$

A more general inequality can be given as follows:

Theorem 2.7. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$ *with* $a < b$*. If* $|f'|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q \geq 1$*, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{17}$$
$$\leq \frac{(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{3(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{3(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}},$$

for all $x \in [a,b]$.

Proof. Suppose that $q \geq 1$. From Lemma 2.1 and by using the well-known power-mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (18)\\
&\leq \frac{(x-a)^2}{4(b-a)}\left(\int_0^1 t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+x}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(x-a)^2}{4(b-a)}\left(\int_0^1 (1+t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1+t)\left|f'\left(tx+(1-t)\frac{a+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (2-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (2-t)\left|f'\left(t\frac{b+x}{2}+(1-t)x\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{4(b-a)}\left(\int_0^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{b+x}{2}\right)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

for all $x \in [a,b]$.
By making use of the similar computations the proof of the theorem is completed. □

Corollary 2.8. *If all the assumptions of Theorem 2.7 are satisfied and if we choose* $x = \frac{a+b}{2}$*, we get the inequality:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \qquad (19)\\
&\leq \frac{b-a}{32}\left\{\left(\max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&+3\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+3\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\left.+\left(\max\left\{\left|f'(b)\right|^q, \left|f'\left(\frac{a+3b}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Additionally,

1. *If* $\left|f'\right|^q$ *is increasing, then (6) holds.*
2. *If* $\left|f'\right|^q$ *is decreasing, then (7) holds.*

3. Applications to Special Means

Let consider the means for arbitrary real numbers $a, b \in \mathbb{R}$. We denote by

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2}; a, b \in \mathbb{R}.$$
2. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}; a, b \in \mathbb{R}, a, b \neq 0.$$

3. The logarithmic mean:

$$L(a,b) = \frac{\ln|b| - \ln|a|}{b-a}; a,b \in \mathbb{R}, |a| \neq |b|, a,b \neq 0.$$

4. Generalized log-mean:

$$L_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}; a,b \in \mathbb{R}, n \in \mathbb{Z}\setminus\{-1,0\}, a \neq b.$$

Now, it is time to give some applications to special means of real numbers by using the results of Section 2.

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\begin{aligned}
&\left|A^n(a,b) - L_n^n(a,b)\right| \\
&\leq n\left(\frac{b-a}{32}\right)\left[\max\left\{\left|\frac{3a+b}{4}\right|^{n-1}, |a|^{n-1}\right\}\right. \\
&+ 3\max\left\{\left|\frac{3a+b}{4}\right|^{n-1}, \left|\frac{a+b}{2}\right|^{n-1}\right\} \\
&+ 3\max\left\{\left|\frac{a+3b}{4}\right|^{n-1}, \left|\frac{a+b}{2}\right|^{n-1}\right\} \\
&\left.+ \max\left\{\left|\frac{a+3b}{4}\right|^{n-1}, |b|^{n-1}\right\}\right].
\end{aligned} \tag{20}$$

Proof. The assertion follows from Corollary 2.3 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned}
&\left|A^n(a,b) - L_n^n(a,b)\right| \\
&\leq n\frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)}, |a|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right. \\
&+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{3a+b}{4}\right|^{q(n-1)}, \left|\frac{a+b}{2}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\
&+ \left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\
&\left.+ \left(\max\left\{|b|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{21}$$

Proof. The assertion follows from Corollary 2.6 when applied to the function $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.3. *Let $a, b \in \mathbb{R}$, $a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then $q \geq 1$, we have*

$$\begin{aligned}
&\left|A^{n}(a,b) - L_{n}^{n}(a,b)\right| \\
&\leq n\left(\frac{b-a}{32}\right)\left\{\left(\max\left\{|a|^{q(n-1)}, \left|\frac{3a+b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right. \\
&+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{3a+b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\
&+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{|b|^{q(n-1)}, \left|\frac{a+3b}{4}\right|^{q(n-1)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{22}$$

Proof. The assertion follows from Corollary 2.8 when applied to the function $f(x) = x^{n}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$. □

Proposition 3.4. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then*

$$\begin{aligned}
&\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \\
&\leq \frac{b-a}{32}\left[\max\left\{\left|\frac{3a+b}{4}\right|^{-2}, |a|^{-2}\right\} + 3\max\left\{\left|\frac{3a+b}{4}\right|^{-2}, \left|\frac{a+b}{2}\right|^{-2}\right\}\right. \\
&\left.+3\max\left\{\left|\frac{a+3b}{4}\right|^{-2}, \left|\frac{a+b}{2}\right|^{-2}\right\} + \max\left\{\left|\frac{a+3b}{4}\right|^{-2}, |b|^{-2}\right\}\right].
\end{aligned} \tag{23}$$

Proof. It is a direct consequence of Corollary 2.3 when applied to the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 3.5. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$, then for all $p > 1$, we have*

$$\begin{aligned}
&\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \\
&\leq \frac{b-a}{16(p+1)^{\frac{1}{p}}}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, |a|^{-2q}\right\}\right)^{\frac{1}{q}}\right. \\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, \left|\frac{a+b}{2}\right|^{-2q(n-1)}\right\}\right)^{\frac{1}{q}} \\
&+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{|b|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{24}$$

Proof. It follows directly from Corollary 2.6 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

Proposition 3.6. *Let $a, b \in \mathbb{R}$, $a < b$, $0 \notin [a, b]$. Then for all $q \geq 1$, we have the inequality*

$$\left|A^{-1}(a,b) - L^{-1}(a,b)\right| \tag{25}$$
$$\leq \frac{b-a}{32}\left\{\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, |a|^{-2q}\right\}\right)^{\frac{1}{q}}\right.$$
$$+3\left(\max\left\{\left|\frac{3a+b}{4}\right|^{-2q}, \left|\frac{a+b}{2}\right|^{-2q}\right\}\right)^{\frac{1}{q}}$$
$$+3\left(\max\left\{\left|\frac{a+b}{2}\right|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}$$
$$\left.+\left(\max\left\{|b|^{-2q}, \left|\frac{a+3b}{4}\right|^{-2q}\right\}\right)^{\frac{1}{q}}\right\}.$$

Proof. It follows directly from Corollary 2.8 for the function, $f(x) = \frac{1}{x}$, $x \in [a, b] \setminus \{0\}$. □

4. Application to the Midpoint Formula

Let d be a division of the interval $[a, b]$, *i.e.* $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$. Let consider the quadrature formulae

$$\int_a^b f(x)dx = M(f, d) + E(f, d),$$

where

$$M(f, d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right)$$

is the midpoint version and the approximation error $E(f, d)$ of the integral $\int_a^b f(x)dx$. The midpoint formula satisfy

$$\left|E(f, d)\right| \leq \frac{K}{24}\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3. \tag{26}$$

If f is not twice differentiable (or the second derivative of f is not bounded on (a, b) then (26) cannot be applied. Following results give some new estimates for the sum of remainders $E(f, d)$in terms of the first derivative.

Proposition 4.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|$ is quasi-convex on $[a, b]$, then for every division d of $[a, b]$, we have:*

$$\left|E(f, d)\right| \tag{27}$$
$$\leq \frac{1}{32}\sum_{i=0}^{n-1} (x_{i+1} - x_i)\left[\max\left\{\left|f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|, \left|f'(x_i)\right|\right\}\right.$$
$$+ 3\max\left\{\left|f'\left(\frac{3x_i + x_{i+1}}{4}\right)\right|, \left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|\right\}$$
$$+ 3\max\left\{\left|f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|, \left|f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|\right\}$$
$$\left.+ \max\left\{\left|f'\left(\frac{x_i + 3x_{i+1}}{4}\right)\right|, \left|f'(x_{i+1})\right|\right\}\right].$$

Proof. By applying Corollary 2.3 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\left| f\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)\,dx \right| \tag{28}$$
$$\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\max\left\{ \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right|, \left| f'(x_i) \right| \right\} \right.$$
$$+ 3 \max\left\{ \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right|, \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right| \right\}$$
$$+ 3 \max\left\{ \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right|, \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right| \right\}$$
$$\left. + \max\left\{ \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right|, \left| f'(x_{i+1}) \right| \right\} \right]$$

which completes the proof. □

Corollary 4.2. *Suppose all the assumptions of Proposition 4.1 are satisfied. Additionally,*

1. *If $\left|f'\right|$ is increasing, then*

$$\left| E(f, d) \right| \tag{29}$$
$$\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right| \right.$$
$$\left. + 3 \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right| + 3 \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right| + \left| f'(x_{i+1}) \right| \right].$$

2. *If $\left|f'\right|$ is decreasing, then*

$$\left| E(f, d) \right| \tag{30}$$
$$\leq \frac{1}{32} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\left| f'(x_i) \right| + 3 \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right| \right.$$
$$\left. + 3 \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right| + \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right| \right].$$

Proposition 4.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is quasi-convex on $[a, b]$ for some fixed $q > 1$, then for every division d of $[a, b]$, we have*

$$\left| E(f, d) \right| \tag{31}$$
$$\leq \frac{1}{16 (p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left\{ \left(\max\left\{ \left| f'(x_i) \right|^q, \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right|^q \right\} \right)^{\frac{1}{q}} \right.$$
$$+ \left(2^{p+1} - 1\right)^{\frac{1}{p}} \left(\max\left\{ \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q, \left| f'\left(\frac{3x_i + x_{i+1}}{4}\right) \right|^q \right\} \right)^{\frac{1}{q}}$$
$$+ \left(2^{p+1} - 1\right)^{\frac{1}{p}} \left(\max\left\{ \left| f'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q, \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right|^q \right\} \right)^{\frac{1}{q}}$$
$$\left. + \left(\max\left\{ \left| f'(x_{i+1}) \right|^q, \left| f'\left(\frac{x_i + 3x_{i+1}}{4}\right) \right|^q \right\} \right)^{\frac{1}{q}} \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is similar to the proof of Proposition 4.1, by applying similar argument to the Corollary 2.6. □

Corollary 4.4. *Suppose all the conditions of Proposition 4.3 are satisfied. Additionally,*

1. *If* $\left|f'\right|^q$ *is increasing, then*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{16(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i) \\ &\times\left\{\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right| + \left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right. \\ &\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right| + \left|f'(x_{i+1})\right|\right\}.\end{aligned} \tag{32}$$

2. *If* $\left|f'\right|^q$ *is decreasing, then*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{16(p+1)^{\frac{1}{p}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left\{\left|f'(x_i)\right| + \left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|\right. \\ &\left.+\left(2^{p+1}-1\right)^{\frac{1}{p}}\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right\}.\end{aligned} \tag{33}$$

Proposition 4.5. *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is quasi-convex on* $[a,b]$ *for some fixed* $q \geq 1$, *then for every division* d *of* $[a,b]$, *we have*

$$\begin{aligned}&\left|E(f,d)\right| \\ &\leq \frac{1}{32}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left\{\left(\max\left\{\left|f'(x_i)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &+3\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{3x_i+x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &+3\left(\max\left\{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\ &\left.+\left(\max\left\{\left|f'(x_{i+1})\right|^q,\left|f'\left(\frac{x_i+3x_{i+1}}{4}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{34}$$

Proof. The proof is similar to the proof of Proposition 4.1, now by applying to Corollary 2.8. □

Corollary 4.6. *Under the assumptions of Proposition 4.5, if*

1. $\left|f'\right|^q$ *is increasing, then (29) holds.*
2. $\left|f'\right|^q$ *is decreasing, then (30) holds.*

5. APPLICATIONS FOR P.D.F's

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=\Pr(X\leq x)=\int_a^x f(t)dt$.

Theorem 5.1. *Under the assumptions of Theorem 2.2, we have the inequality;*

$$\begin{aligned}
&\left|\Pr(X\leq x)-\frac{1}{b-a}\left(b-E(x)\right)\right| \\
&\leq \frac{(x-a)^2}{8(b-a)}\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|,\left|f'(a)\right|\right\} \\
&+\frac{3(x-a)^2}{8(b-a)}\max\left\{\left|f'(x)\right|,\left|f'\left(\frac{a+x}{2}\right)\right|\right\} \\
&+\frac{3(b-x)^2}{8(b-a)}\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|,\left|f'(x)\right|\right\} \\
&+\frac{(b-x)^2}{8(b-a)}\max\left\{\left|f'(b)\right|,\left|f'\left(\frac{b+x}{2}\right)\right|\right\},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. The proof is immediate follows from the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt.$$

□

Theorem 5.2. *Under the assumptions of Theorem 2.5, we have the inequality;*

$$\begin{aligned}
&\left|\Pr(X\leq x)-\frac{1}{b-a}\left(b-E(x)\right)\right| \\
&\leq \frac{1}{4(b-a)(p+1)^{\frac{1}{p}}}\left\{(x-a)^2\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&+(x-a)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(x)\right|^q,\left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+(b-x)^2\left(2^{p+1}-1\right)^{\frac{1}{p}}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q,\left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\left.+(b-x)^2\left(\max\left\{\left|f'(b)\right|^q,\left|f'\left(\frac{b+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. Likewise the proof of the previous theorem, by using the fact that;

$$E(x)=\int_a^b tdF(t)=b-\int_a^b F(t)\,dt$$

the proof is completed. □

Theorem 5.3. *Under the assumptions of Theorem 2.7, we have inequality;*

$$\begin{aligned}
&\left|\Pr(X \leq x) - \frac{1}{b-a}(b - E(x))\right| \\
&\leq \frac{(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{a+x}{2}\right)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{3(x-a)^2}{8(b-a)}\left(\max\left\{\left|f'(x)\right|^q, \left|f'\left(\frac{a+x}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{3(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q, \left|f'(x)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{8(b-a)}\left(\max\left\{\left|f'\left(\frac{b+x}{2}\right)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}$$

where $E(x)$ is the expectation of X.

Proof. The proof is similar to the proof of the previous result. □

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