New Refinements of Hadamard Integral Inequality via k-Fractional Integrals for P- Convex Function

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Abstract. In this study, we use k-fractional integrals to establish some new integral inequalities for p-convex function. These integral inequalities includes some new estimations for Hadamard inequality via k-fractional integrals.

1. Introduction

A function $\rho[\varepsilon, \epsilon'] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $u, v \in [\varepsilon, \epsilon']$ and $t \in [0, 1]$, the following inequality holds:

$$\rho(tu + (1 - t)v) \le t\rho(u) + (1 - t)\rho(v).$$

We say that ρ is concave if $(-\rho)$ is convex. If ρ is both convex and concave, then ρ is to be said affin function. The affine functions are in the form $\varepsilon_1 u + \epsilon'_1$, for suitable constants $\varepsilon_1, \epsilon'_1$.

This definition has its origins in Jensen's results and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

The following double inequality is well known in the literature as Hadamard's inequality:
Let $\rho : [\varepsilon, \epsilon'] \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an subinterval of real numbers, $\eta, \eta' \in [\varepsilon, \epsilon']$ and $\eta < \eta'$, we have

$$\rho\left(\frac{\eta + \eta'}{2}\right) \le \frac{1}{\eta' - \eta} \int_{\eta}^{\eta'} \rho(u)dx \le \frac{\rho(\eta) + \rho(\eta')}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if ρ is concave. In [7], there are many inequalities associated with (1.1) for different function types.

The definition and basic elements about the subject are following.

Definition 1.1. *[12] Let $\rho \in L_1[\varepsilon, \epsilon']$. The Riemann Liouville integrals $J^\alpha_{\varepsilon^+} f$ and $J^\alpha_{\epsilon'-} f$ of order $\alpha > 0$ with $\varepsilon \geq 0$ are defined by*

$$J^\alpha_{\varepsilon^+} f \ (u) = \frac{1}{\Gamma(\alpha)} \int_\varepsilon^u (u - t)^{\alpha-1} f(t) \, dt \ , \qquad u > \varepsilon$$

and

$$J^\alpha_{\epsilon'-} f \ (u) = \frac{1}{\Gamma(\alpha)} \int_u^{\epsilon'} (t - u)^{\alpha-1} f(t) \, dt \ , \qquad u < \epsilon'$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$. Here is $J^0_{\varepsilon^+} f(u) = J^0_{\epsilon'-} f(u) = f(u)$.

In the case of $\alpha = 1$ the fractional integral reduces to the classical integral.

Definition 1.2. *[12] Let $\rho \in L_1[\varepsilon, \epsilon']$. The right and the left $k-$Riemann Liouville integrals $J^{\frac{\alpha}{k}-1}_{\varepsilon^+} \rho$ and $J^{\frac{\alpha}{k}-1}_{\epsilon'-} \rho$ of order $\alpha > 0$, $k > 0$ with $\varepsilon > 0$ are defined by*

$$J^{\alpha,k}_{a^+} \rho(u) = \frac{1}{k\Gamma_k(\alpha)} \int_\varepsilon^u (u - t)^{\frac{\alpha}{k}-1} \rho(t) \, dt \ , \qquad u > \varepsilon$$

and

$$J^{\alpha,k}_{\epsilon'-} \rho(u) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\epsilon'} (t - u)^{\frac{\alpha}{k}-1} \rho(t) \, dt \ , \qquad u < \epsilon'$$

Definition 1.3. *[7] We say that $\rho: I \to \mathbb{R}$ is a P-function, or that f belongs to the class P(I), if ρ is a non-negative function and for all $u, v \in I$, $t \in [0, 1]$, we have*

$$\rho(tu + (1 - t)v) \leq \rho(u) + \rho(v).$$

P(I) contain all nonnegative monotone convex and quasi convex functions.

Definition 1.4. *[12] Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$: The function f is said to be quasi-convex on I if inequality*

$$\rho(tu + (1 - t)v) \leq \sup\{\rho(u); \rho(v)\} \qquad (QC)$$

holds for all $u; v \in I$ and $t \in [0; 1]$

In [14], M.Zeki Sarıkaya et.all proved the following inequality with connected (1.1) for fractional integrals using the definition of convexity:

$$f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[J^\alpha_{a^+} f(b) + J^\alpha_{b^+} f(a)\right] \leq \frac{f(a) + f(b)}{2} \tag{2}$$

The aim of this paper is to rewrite inequality written in type (1.2) for the Riemann-Liouville $k-$fractional, using the P-convex function. In a way, it is a continuation of my previous works. see[12]

In [12] , we obtained the following lemma for $k-$Riemann Liouville fractional integrals.

Lemma 1.5. *Let $\rho: I \subset \mathbb{R} \to \mathbb{R}$ be a function on I ,where $\eta, \eta' \in I$ with $t \in [0, 1]$. If $\rho \in L[\eta, \eta']$, then for all $\eta \leq u < v \leq \eta'$ and $\alpha > 0$ we have:*

$$\frac{\rho(u) + \rho(v)}{v - u} + \frac{\alpha\Gamma_k(\alpha)}{(v - u)^{\frac{\alpha}{k}-1}} \left[I^{\alpha,k}_{u^+} \rho(v) + I^{\alpha,k}_{v^-} \rho(u)\right] \tag{3}$$

$$= \int_0^1 (1 - t)^{\frac{\alpha}{k}} \rho'(tu + (1 - t)v) \, dt$$

$$+ \int_0^1 (1 - t)^{\frac{\alpha}{k}} \rho'((1 - t)u + tv) \, dt.$$

for each $u, v \in [\eta, \eta']$.

Recently, Mathematicians have been published a huge amount papers for fractional order operators using the various convex functions, see [1–6, 8–11, 13, 15–18]

2. MAIN RESULTS

Theorem 2.1. *Let $\rho: S \subset \mathbb{R} \to \mathbb{R}$ be a function on I ,where $\eta, \eta' \in S$ with $t \in [0, 1]$. If $\rho' \in L[\eta, \eta']$,*

for all $\eta \le u < v \le \eta'$ and $\alpha, k > 0$. If ρ' is $p-$convex on $[u, v]$. Then we have the inequality

$$\left| \frac{\rho(v) + \rho(u)}{v - u} + \frac{\alpha \Gamma_k(\alpha)}{(v - u)^{\frac{\alpha}{k} - 1}} \left[I_{u^+}^{\alpha,k} \rho(v) + I_{v^-}^{\alpha,k} \rho(v) \right] \right| \le 2 \frac{k}{\alpha + k} \left[\left| \rho'(u) + \rho'(v) \right| \right]. \tag{4}$$

Proof. By using properties modulus and the identity in (1.3) with the p convexity of ρ'

$$\left| \frac{\rho(v) + \rho(u)}{v - u} + \frac{\alpha \Gamma_k(\alpha)}{(v - u)^{\frac{\alpha}{k} - 1}} \left[I_{u^+}^{\alpha,k} \rho(v) + I_{v^-}^{\alpha,k} \rho(v) \right] \right|$$

$$\le \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(tu + (1 - t)v) \right| dt + \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'((1 - t)u + tv) \right| dt.$$

$$J_1 = \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(tu + (1 - t)v) \right| dt$$

$$\le \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left[\left| \rho'(u) \right| + \left| \rho'(v) \right| \right] dt = \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(u) \right| dt + \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(v) \right| dt$$

$$= \left[\left| \rho'(u) \right| + \left| \rho'(v) \right| \right] \frac{k}{\alpha + k}$$

□

and

$$J_2 = \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'((1 - t)u + tv) \right| dt$$

$$\le \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left[\left| \rho'(u) \right| + \left| \rho'(v) \right| \right] dt = \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(u) \right| dt + \int_0^1 (1 - t)^{\frac{\alpha}{k}} \left| \rho'(v) \right| dt$$

$$= \left[\left| \rho'(u) \right| + \left| \rho'(v) \right| \right] \frac{k}{\alpha + k}$$

Then adding J_1 and J_2 we get the (2.1) inequality.

Corollary 2.2. *when $\alpha = k = 1$ in (2.1) we obtain the inequality*

$$\left| \frac{\rho(v) + \rho(u)}{2(v - u)} + \int_u^v f(t) dt \right| \le \left[\left| \rho'(u) + \rho'(v) \right| \right].$$

Theorem 2.3. *Let $\rho: I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I ,where $\eta, \eta' \in I$ with $t \in [0, 1]$. If $\rho' \in L[\eta, \eta']$, for all $\eta \le u < v \le \eta'$ and $\alpha, k > 0$. If $\left| \rho' \right|^q$ is $p-$convex on $[u, v]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ Then we have the*

inequality

$$|J| \leq 2\frac{k}{\alpha+k}\left[\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right]^{\frac{1}{q}}. \tag{5}$$

where

$$J = \frac{\rho(v)+\rho(u)}{v-u} + \frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v) + I_{v^-}^{\alpha,k}\rho(v)\right]$$

Proof. If we use the lemma (1.3) in view of the properties of modulus and Power Mean inequality with $p-$convex of $\left|\rho'\right|^q$ on $[u,v]$, we have

$$
\begin{aligned}
|J_1| &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|dt \\
&\leq \left(\int_0^1 (1-t)^{\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\int_0^1 (1-t)^{\frac{\alpha}{k}}dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(u)\right|^q dt + \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'(v)\right|^q dt\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left[\left(\frac{\alpha}{\alpha+k}\right)\left|\rho'(u)\right|^q + \left(\frac{\alpha}{\alpha+k}\right)\left|\rho'(v)\right|^q\right]^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left[\left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)\right]^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{p}}\left(\frac{\alpha}{\alpha+k}\right)^{\frac{1}{q}}\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
J_2 &= \int_0^1 (1-t)^{\frac{\alpha}{k}}\left|\rho'((1-t)u+tv)\right|dt \\
&\leq \left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
$$

□

Now, then we obtain

$$|J| \leq |J_1| + |J_2| = 2\left(\frac{\alpha}{\alpha+k}\right)\left(\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right)^{\frac{1}{q}}$$

which proof the inequality (2.2).

Theorem 2.4. *Let $\rho:[\eta,\eta'] \to \mathbb{R}$ be a differentiable mapping on $[\eta,\eta']$,where $\eta < \eta'$ such that $\rho' \in L[\eta,\eta']$ If $\left|\rho'\right|^q$ is $p-$convex on $[u,v]$ and $\eta \leq u < v \leq \eta'$ and . $p > 1$,with $t \in [0,1]$. Then we have the*

$$\left|\frac{\rho(v)+\rho(u)}{v-u} + \frac{\alpha\Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}}\left[I_{u^+}^{\alpha,k}\rho(v) + I_{v^-}^{\alpha,k}\rho(v)\right]\right| \tag{6}$$

$$\leq 2\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left[\left|\rho'(u)\right|^q + \left|\rho'(v)\right|^q\right]^{\frac{1}{q}}.$$

where $\frac{1}{p} + \frac{1}{q} = 1, \quad k > 0, \quad \alpha > 1.$

Proof. By using the identity that is given in (1.3) with classic hölder inequaliy for each term and the definition p−convex of $|\rho'|$, we have

$$\left| \frac{\rho(v) + \rho(u)}{v - u} + \frac{\alpha \Gamma_k(\alpha)}{(v-u)^{\frac{\alpha}{k}-1}} \left[I_{u^+}^{\alpha,k} \rho(v) + I_{v^-}^{\alpha,k} \rho(v) \right] \right|$$

$$\leq \left(\int_0^1 (1-t)^{(\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |\rho'(tu + (1-t)v)|^q dt \right)^{\frac{1}{q}}$$

$$+ \left(\int_0^1 (1-t)^{(\frac{\alpha}{k})p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |\rho'((1-t)u + tv)|^q dt \right)^{\frac{1}{q}}$$

$$\leq 2 \left(\frac{k}{k + \alpha p} \right)^{\frac{1}{p}} \left[|\rho'(u)|^q + |\rho'(v)|^q \right]^{\frac{1}{q}}.$$

□

which proof the inequality (2.3) .

Corollary 2.5. *Under conditions of Theorem 3 we have*

$$\left| \frac{\rho(v) + \rho(u)}{2(v-u)} + \frac{\alpha \Gamma_k(\alpha)}{2(v-u)^{\frac{\alpha}{k}-1}} \left[I_{u^+}^{\alpha,k} \rho(v) + I_{v^-}^{\alpha,k} \rho(v) \right] \right| \leq \left[|\rho'(u)|^q + |\rho'(v)|^q \right]^{\frac{1}{q}}$$

Proof. Since $\lim\limits_{p \to \infty} 2 \left(\frac{k}{\alpha p + k} \right)^{\frac{1}{p}} = 2$ the result is clear. □

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