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The Category of Soft Topological Hyperrings

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Abstract. In this study, which focuses on the intersection of soft set theory and topological hyperrings, the concept of soft topological hyperrings is proposed and its relation with topological hyperrings is examined. Morever, some characterizations related to the family of soft topological hyperrings are obtained and the category of soft topological hyperrings is established. Finally, the concept of soft topological subhyperrings is described and several structural features are studied.

1. Introduction

Due to its compelling structure, the literature on ring theory has received less attention than group theory, but it has gained enough attention lately. [27] can be given as an example to the studies in this area. More interesting is the relationship between soft set theory and hyper structure of group/ring theory. The algebraic hyperstructures that emerged with the introduction of hypergroups by Marty are considered as a generalization of classical algebraic structures [1]. The concept of hyperrings plays important role in the the theory of algebraic hyperstructures. Hyperrings, defined by M. Krasner, have been studied by various researchers [19-21].In particular, the book "Hyperrings Theory and Applications" is a good review resource on this topic [18]. Although there are many algebraic studies on hyperrings, topological studies on them are very limited. By defining the concept of topological hyperrings, Nodehi et al. investigated some differences between the topological rings and topological hyperrings [20].

One of the fertile areas for many researchers working on theories modeling uncertainty is the soft set theory initiated by Molodstov, since it has many applications in economics, computer science, biology, engineering, environment, social science and medical science [2]. The easy applicability of this theory in other fields of mathematics, especially algebra and topology, has enabled many important studies. Firstly, Maji et al. developed the application of soft set theory in decision making problems and introduced some operations on soft sets [23]. Aktas and cagman presented the definition of soft groups and studied their fundamental operations [5]. Other algebraic works of soft set theory can be found [4, 6]. On the other hand, topological studies on soft sets were first put forth by Shabir and Naz [8]. By defining the notion of a soft topological space, they examined the separation axioms in a soft topological space. For more details, see [7, 9-11,13].

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Recently, some researchers have applied soft set theory to algebraic hyperstructures. Yamak, the first of these, developed the concepts of soft hypergroupoids and soft subhypergroupoids [12]. Later on, Wang et al. introduced the concepts of soft (normal) polygroups and soft (normal) subpolygroups [16]. Selvachandran and Salleh studied the concepts of soft hypergroups and soft subhypergroups as an expansion of soft hypergroupoids and soft subhypergroupoids [15]. Also, Selvachandran presented the definitions of Soft hyperrings and soft hyperring homomorphism [17]. In [14] Oguz developed the concept of soft topological polygroups by examining polygroups, an important subclass of hypergroups, with a soft topological approach.

The main purpose of this study is to present the concept of soft topological hyperrings by examining hyperrings which is one of the the algebraic hyperystructures with soft set theory from the topological point of view. Further, the relation between soft topological hyperrings and soft hyperrings is investigated and several theoretical results are given. By defining the concept of soft topological hyperring homomorphism, the category of soft topological hyperrings is establıshed. This study completed by giving the definition of soft topological subhyperrings and examining some relevant properties.

2. Preliminaries

In this section, some notions and results about soft sets, topological hyperrings and soft hyperrings to be used in the sequel will be presented.

Let X be an initial universe set and E be a set of parameters. Also, let $P(X)$ denotes the power set of X and $A \subset E$. The definition of a soft set introduced by Molodtsov is as follows:

Definition 2.1. [2] *A pair $(\mathcal{F}, A)$ is called a soft set over X, where $\mathcal{F}$ is a mapping defined by*

$$\mathcal{F} : A \longrightarrow P(X)$$

Note that a soft set over X is actually a parametrized family of subsets of the universe X.

Definition 2.2. [3] *Let $(\mathcal{F}, A)$ and $(\mathcal{G}, B)$ be two soft sets over the common universe X. Then, $(\mathcal{F}, A)$ is called a soft subset of $(\mathcal{G}, B)$ (i.e., $(\mathcal{F}, A)\widetilde{\subset}(\mathcal{G}, B)$) if*
i. *$A \subseteq B$,*
ii. *$\mathcal{F}(\varepsilon)$ and $\mathcal{G}(\varepsilon)$ are identical approximations for all $\varepsilon \in A$.*

Definition 2.3. [12] *The support of a soft set (F, A) is defined as a set*

$$Supp(\mathcal{F}, A) = \{\varepsilon \in A : F(\varepsilon) \neq \emptyset\}$$

If $Supp(\mathcal{F}, A)$ is not equal to the empty set, then $(\mathcal{F}, A)$ is called non-null.

In the following, some generalizations are given for the nonempty family $\{(\mathcal{F}_\alpha, A_\alpha)|\alpha \in I\}$ of soft sets over the common universe X.

Definition 2.4. [24] *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha)\ |\alpha \in I\}$ is defined by a soft set $(\mathcal{F}, A) = \widetilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha)$ such that $A = \bigcap_{\alpha\in I} A_i \neq \emptyset$ and $\mathcal{F}(\alpha) = \bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A_\alpha$.*

Definition 2.5. [24] *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha)|\alpha \in I\}$ is a soft set $(\mathcal{F}, A) = (\bigcap_{\mathcal{E}})_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha)$ such that $A = \bigcup_{\alpha\in I} A_\alpha$ and $\mathcal{F}(\varepsilon) = \bigcap_{\alpha\in I(\varepsilon)} \mathcal{F}_\alpha(\varepsilon)$, $I(\varepsilon) = \{\alpha \in I|\varepsilon \in A_\alpha\}$ for all $\varepsilon \in A_\alpha$.*

Definition 2.6. [24] *The $\wedge-$intersection of the family $\{(F_\alpha, A_\alpha)|\alpha \in I\}$ is defined by a soft set $(F, A) = \widetilde{\bigwedge}_{\alpha\in I}(F_\alpha, A_\alpha)$ such that $A = \Pi_{\alpha\in I}A_\alpha$ and $F((\varepsilon_\alpha)_{\alpha\in I}) = \bigcap_{\alpha\in I} F_\alpha(\varepsilon_\alpha)$ for all $(\varepsilon_\alpha)_{\alpha\in I} \in A_\alpha$.*

Definition 2.7. [26] *Let $\mathcal{R}$ be a non-empty set and $P^*(\mathcal{R})$ denote the family of non-empty subsets of $\mathcal{R}$. Then, the mapping $\cdot : \mathcal{R} \times \mathcal{R} \longrightarrow P^*(\mathcal{R})$ is called a hyperoperation and the pair $(\mathcal{R}, \cdot)$ is also called hypergroupoid.*

Definition 2.8. [26] *A hypergroup is a hypergroupoid* $(\mathcal{R}, \cdot)$ *which satisfies the following axioms:*

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *for all* $x, y, z \in \mathcal{R}$

(ii) $x \cdot \mathcal{R} = \mathcal{R} \cdot x$ *for all* $x \in \mathcal{R}$.

A semi hypergroup is a hypergroupoid $(\mathcal{R}, \cdot)$ if for all $x, y, z \in \mathcal{R}$, we have $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Now, the definitions of topological hyperring, soft topological ring and soft hyperring will be recalled.

Definition 2.9. [18] *Let* $(\mathcal{F}, A)$ *be a non-null soft set on a commutative ring* R *endowed with the topology* τ. *Then, the triplet* $(\mathcal{F}, A, \tau)$ *is called a soft topological ring over* R *if the following conditions hold for all* $\varepsilon \in A$:
i. $\mathcal{F}(\varepsilon)$ *is a subring of* G *for all* $\varepsilon \in A$.
ii. *the mapping* $F(\varepsilon) \times F(\varepsilon) \longrightarrow F(\varepsilon)$ *defined by* $(x, y) \longmapsto x - y$ *is continuous.*
iii. *the mapping* $F(\varepsilon) \times F(\varepsilon) \longrightarrow F(\varepsilon)$ *defined by* $(x, y) \longmapsto x \cdot y$ *is continuous.*

Definition 2.10. [22] *A hyperring is an algebraic system* $(\mathcal{R}, +, \cdot)$ *which satisfies the following axioms:*
i. $(\mathcal{R}, +)$ *is a commutative hypergroup.*
ii. $(\mathcal{R}, \cdot)$ *is a semihypergroup.*
iii. *The hyperoperation "$\cdot$" is distributive with respect to the hyperoperation "$+$".*

Example 2.11. [22] *Let* $\mathcal{R} = \{0, 1\}$ *be a set with two hyperoperations defined as follows:*

$+$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

$\cdot$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

So it can be easily verified that $(\mathcal{R}, +, \cdot)$ *is a hyperring.*

Definition 2.12. [25] *A non-empty subset* $\mathcal{R}'$ *of a hyperring* $(\mathcal{R}, +, \cdot)$ *is said to be a subhyperring of* $\mathcal{R}$ *if* $(\mathcal{R}', +, \cdot)$ *itself is a hyperring.*

Definition 2.13. [20] *Let* $(\mathcal{R}, \tau)$ *be a topological space and* $P^*(\mathcal{R})$ *denote the family of non-empty subsets of* $\mathcal{R}$. *Then, the collection* $\mathcal{B}$ *consisting of all sets* $\mathcal{S}_V = \{U \in P^*(\mathcal{R}) : U \subseteq V, U \in \tau\}$ *is a base for a topology on* $P^*(\mathcal{R})$ *denoted by* τ^*.

Definition 2.14. [20] *Let* $(\mathcal{R}, +, \cdot)$ *be a hyperring and* $(\mathcal{R}, \tau)$ *be a topological space. Then, algebraic hyperstructure* $(\mathcal{R}, +, \cdot, \tau)$ *is called a topological hyperring if three hyperoperations " $+$ ", " $\cdot$ " and "$/$" are continuous.*

Remark 2.15. [20] *Every topological ring is a topological hyperring by trivial hyperoperations.*

Definition 2.16. [17] *Let* $(\mathcal{F}, A)$ *be a non-null soft set over the hyperring* R. *Then the pair* (F, A) *is said to be a soft hyperring over* R *if* $\mathcal{F}(\varepsilon)$ *is a subhyperring of* R *for all* $\varepsilon \in Supp(\mathcal{F}, A)$.

Example 2.17. [25] *Consider a hyperring* $(\mathcal{R}, +, \cdot)$ *with the hyperoperations as follows:*

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	0	0	0

Define a soft set (F, A) *over* $\mathcal{R} = \{0, 1, 2, 3\}$, *where* $A = \mathcal{R}$, *by* $F(0) = \{0, 2\}$, $F(1) = \{0, 3\}$, $F(2) = \{0\}$ *and* $F(3) = \{0, 1\}$. *Then it is clear that* $F(0), F(1), F(2)$ *and* $F(3)$ *are subhyperrings of* $\mathcal{R}$. *Thus* (F, A) *is a soft hyperring over* $\mathcal{R}$.

3. Soft Topological Hyperrings

In this section, the concept of soft topological hyperrings will be introduced and some important characterizations of them will be established. By presenting the concept of soft topological subrings, the related structural properties will also be examined.

Definition 3.1. *Let τ be a topology on the hyperring $\mathcal{R}$. Let $\mathcal{F} : A \longrightarrow P(\mathcal{R})$ be a mapping, where $P(\mathcal{R})$ is the set of all subhyperrings of $\mathcal{R}$, and A is the set of parameters. The system $(\mathcal{F}, A, \tau)$ is called a soft topological hyperring over $\mathcal{R}$ if the following statements hold for all $\varepsilon \in Supp(\mathcal{F}, A)$:*
i. *$F(\varepsilon)$ is a subhyperring of $\mathcal{R}$;*
ii. *Three hyperoperations $+, \cdot, / : \mathcal{F}(\varepsilon) \times \mathcal{F}(\varepsilon) \longrightarrow P^*(\mathcal{F}(\varepsilon))$ are continuous with respect to the topologies induced by $\tau \times \tau$ and τ^*.*

A trivial verification shows that if $\mathcal{R}$ is a topological hyperring, it is sufficient to hold only the statement *i.* in the above definition in order to called the system $(\mathcal{F}, A, \tau)$ as a soft topological hyperring. Besides, the soft topological hyperring $(\mathcal{F}, A, \tau)$ can be considered as a parameterized family of subhyperrings of the topological hyperring $\mathcal{R}$.

Example 3.2. *Every soft topological ring is a soft topological hyperring.*

Example 3.3. *Consider the hyperring $\mathbb{R}$ of real numbers with its natural topology τ such that the hyperoperations $x + y = x \cdot y = \{x, y\}$ for all $x, y \in \mathbb{R}$. Suppose $A = \mathbb{N}$. Then for all $\varepsilon \in A$, the mapping F is defined as*

$$\begin{array}{rcl} F : \mathbb{N} & \longrightarrow & P^*(\mathbb{R}) \\ \varepsilon & \mapsto & F(\varepsilon) = \begin{cases} \{0, \varepsilon\} & \varepsilon \text{ tek} \\ \mathbb{Q} & \varepsilon \text{ çift} \end{cases} \end{array}$$

In either case, it can be clearly checked that $F(\varepsilon)$ is a subhyperring of the topological hyperring $\mathbb{R}$. Hence, the triplet $(\mathcal{F}, A, \tau)$ a soft topological hyperring over $\mathbb{R}$.

Definition 3.4. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{R}$. Then $(\mathcal{F}, A, \tau)$ is said to be*
i. *an identity soft topological hyperring if $F(\varepsilon) = \emptyset$ for all $\varepsilon \in A$.*
ii. *an absolute soft topological hyperring if $F(\varepsilon) = \mathcal{R}$ for all $\varepsilon \in A$.*

Example 3.5. *In the example above, assuming $A = \mathbb{R}$ and $F(\varepsilon) = \{\omega \in \mathbb{R} : \varepsilon + \omega = \{\varepsilon\}\}$ for all $\varepsilon \in A$, it is easily obtained that $(\mathcal{F}, A, \tau)$ is an identity soft topological hyperring over $\mathbb{R}$.*

In the following, we present the relationship between soft hyperrings and soft topological hyperrings.

Theorem 3.6. *Every soft hyperring on a topological hyperring $\mathcal{R}$ is a soft topological hyperring.*

Proof. Consider a soft hyperring $(\mathcal{F}, A)$ over the topological hyperring $\mathcal{R}$ with the topology τ. Since $\mathcal{F}(\varepsilon)$ is a subhyperring of $\mathcal{R}$ for all $\varepsilon \in A$, $\mathcal{F}(\varepsilon)$ is also a topological subhyperring of $\mathcal{R}$ with recpect to the topologies induced by τ and τ^* for all $\varepsilon \in A$. Thus, $(\mathcal{F}, A, \tau)$ is a soft topological hyperhyperring over $\mathcal{R}$. □

Remark 3.7. *Each soft hyperring $\mathcal{R}$ can be transformed into a soft topological hyperring by equipping both $\mathcal{R}$ and $P^*(\mathcal{R})$ with discrete or indiscrete topology. But the converse of this statement is not true, meaning that every soft hyperring over a hyperring is not a soft topological hyperring.*

Some generalizations for a nonempty family of soft topological hyperrings are introduced here:

Theorem 3.8. *Let $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological hyperrings over $\mathcal{R}$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ with $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*
ii. *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha) | \alpha \in I\}$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*
iii. *The $\wedge-$intersection $\widetilde{\bigwedge}_{\alpha \in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ is a soft topological hypergroup over $\mathcal{R}$ if it is non-null*

Proof. **i**. The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ given by the soft set $\widetilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) = (\mathcal{F}, A, \tau)$ such that $\bigcap_{\alpha\in I}\mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A$ from Definition 2.4. Take $\varepsilon \in Supp(\mathcal{F}, A)$. By the assumption, $\bigcap_{\alpha\in I}\mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ such that $\mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ for all $\alpha \in I$. Since $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ is a non-empty family of soft topological hyperrings over $\mathcal{R}$, this implies that $\mathcal{F}_\alpha(\varepsilon)$ is also a topological subhyperring of $\mathcal{R}$ for all $\alpha \in I$. It is then evident that $\bigcap_{\alpha\in I}\mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{R}$. Therefore, $(\mathcal{F}, A, \tau)$ is a soft topological hyperring over $\mathcal{R}$.
ii. The proof is similar to *i*.
iii. Choose $(\mathcal{F}, A, \tau) = \widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ for a non-empty family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) | \alpha \in I\}$ of soft topological hyperrings over $\mathcal{R}$. Let $\varepsilon \in Supp(\mathcal{F}, A)$. It follows from the hypothesis $\bigcap_{\alpha\in I}\mathcal{F}_\alpha(\varepsilon_\alpha) \neq \emptyset$ that $\mathcal{F}_\alpha(\varepsilon_\alpha) \neq \emptyset$ for all $\alpha \in I$ and $(\varepsilon_\alpha)_{\alpha\in I} \in A_\alpha$. Thus, $\mathcal{F}_\alpha(\varepsilon_\alpha)$ is a topological subhyperring of $\mathcal{R}$ for all $\alpha \in I$ so that their intersection must be a topological subhyperring of $\mathcal{R}$ too. Clearly, $(\mathcal{F}, A, \tau)$ is a soft topological hyperring over $\mathcal{R}$. □

3.1. Soft Topological Hyperring Homomorphisms

Definition 3.9. *Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ be soft topological hyperrings over $\mathcal{R}$ and $\mathcal{S}$, respectively. Let $\phi : A \longrightarrow B$ and $\psi : \mathcal{R} \longrightarrow \mathcal{S}$ be two mappings. Then the pair (ψ, ϕ) is called a soft topological homomorphism if the following statements are satisfied:*
i. *ψ is a strong homomorphism;*
ii. *$\psi(\mathcal{F}(\varepsilon)) = \mathcal{K}(\phi(\varepsilon))$ for all $\varepsilon \in Supp(\mathcal{F}, A)$;*
iii. *$\psi_\varepsilon : (\mathcal{F}(\varepsilon), \tau_{\mathcal{F}(\varepsilon)}) \longrightarrow (\mathcal{K}(\phi(\varepsilon)), \tau'_{\mathcal{K}(\phi(\varepsilon))})$ continuous and open for all $\varepsilon \in Supp(\mathcal{F}, A)$.*

Namely, a soft topological homomorphism (ψ, ϕ) is a mapping of soft topological hyperrings. In this direction, we obtain a new category whose objects are soft topological hyperrings and whose arrows are soft topological homomorphisms.

Note that If ψ is a isomorphism, ϕ is bijective, then the pair (ψ, ϕ) is said to be a soft topological isomorphism, and $(\mathcal{F}, A, \tau)$ is soft topologically isomorphic to $(\mathcal{K}, B, \tau')$ denoted by $(\mathcal{F}, A, \tau) \simeq (\mathcal{K}, B, \tau')$.

Example 3.10. *Let $(\mathcal{K}, B, \tau)$ be a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. Then the pair $(\mathcal{I}, i)$ is a soft topological homomorphism from $(\mathcal{K}, B, \tau)$ to $(\mathcal{F}, A, \tau)$, where $i : B \longrightarrow A$ is an inclusion map and $\mathcal{I} : \mathcal{R} \longrightarrow \mathcal{R}$ is an identity map.*

Example 3.11. *Let $(\mathcal{F}, A)$ and $(\mathcal{K}, B)$ be the two soft homomorphic hyperrings defined over $\mathcal{R}$ and $\mathcal{S}$, resp. Then, it is easy to obtain that $(\mathcal{F}, A, \tau)$ is soft topological homomorphic to $(\mathcal{K}, B, \tau)$ such that τ is discrete or anti-discrete topology. So, any soft homomorphic hyperrings can be reviewed as soft topological homomorphic hyperrings with the discrete or anti-discrete topology.*

At the moment, we can easily deduce that

Theorem 3.12. *Let the pair (ψ, ϕ) be a soft topological homomorphism between the soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ defined over $\mathcal{R}$ and $\mathcal{S}$, resp. Then if $\phi : A \longrightarrow B$ be an injective mapping, $(\psi(\mathcal{F}), B, \tau')$ is a soft topological hyperring over $\mathcal{S}$*

Proof. Consider two soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{R}$ and $\mathcal{S}$, respectively. Since $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ is a soft topological homomorphism, it follows that $\phi(Supp(\mathcal{F}, A)) = Supp(\psi(\mathcal{F}), B)$. Consider $b \in Supp(\psi(\mathcal{F}), B)$. So there exist $\varepsilon \in Supp(\mathcal{F}, A)$ such that $\phi(\varepsilon) = b$ and hence $\mathcal{F}(\varepsilon) \neq \emptyset$. Also, it is evident that $\mathcal{F}(\varepsilon)$ is a topological subhyperring of $\mathcal{R}$ and is also a topological hyperring with respect to the topology induced by τ. Since ψ is a strong homomorphism, we obtain that $\psi(\mathcal{F}(\varepsilon))$ is a topological subhyperring of $\mathcal{H}'$ with respect to the topology induced by τ'. Consequently, $(\psi(\mathcal{F}), B, \tau')$ is a soft topological hypergroup over $\mathcal{S}$. □

Theorem 3.13. *Let the pair (ψ, ϕ) be a soft topological homomorphism between the soft topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{H}$ and $\mathcal{H}'$, resp. Then $(\psi^{-1}(\mathcal{K}), A, \tau)$ is a soft topological hyperring over $\mathcal{R}$ if it is non-null.*

Proof. Since the pair (ψ, ϕ) be a soft topological homomorphism, this implies

$$\phi(Supp(\psi^{-1}(\mathcal{K}), A)) = \phi^{-1}(Supp(\mathcal{K}, B))$$

for all $b \in Supp(\mathcal{K}, B)$. When $a \in Supp(\psi^{-1}(\mathcal{K}), A)$, we get $\phi(\varepsilon) \in Supp(\mathcal{K}, B)$. Thus, the nonempty set $\mathcal{K}(\phi(\varepsilon))$ is a topological subhyperring of $\mathcal{H}'$ and is also a topological hyperring with respect to the topology induced by τ'. Since ψ is a strong homomorphism, it follows that $\psi^{-1}(\mathcal{K}(\phi(\varepsilon))) = \psi^{-1}(\mathcal{K}(\varepsilon))$ is a topological subhyperring of $\mathcal{R}$ with respect to the topology induced by τ. This means that $(\psi^{-1}(\mathcal{K}), A, \tau)$ is a soft topological hyperring over $\mathcal{R}$. □

Theorem 3.14. *Let $(\mathcal{F}, A, \tau)$, $(\mathcal{K}, B, \tau')$ and $(\mathcal{N}, C, \tau'')$ be soft topological hyperrings over $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$, respectively. If $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ and $(\psi', \phi') : (\mathcal{K}, B, \tau') \longrightarrow (\mathcal{N}, C, \tau'')$ are two soft topological homomorphisms, then the pair $(\psi' \circ \psi, \phi' \circ \phi)$ is a soft topological homomorphism.*

Proof. Consider two soft topological homomorphisms $(\psi, \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{K}, B, \tau')$ and $(\psi', \phi') : (\mathcal{K}, B, \tau') \longrightarrow (\mathcal{N}, C, \tau'')$. By Definition 3.9, it follows that $\psi : \mathcal{H} \longrightarrow \mathcal{H}'$ and $\psi' : \mathcal{H}' \longrightarrow \mathcal{H}''$ are two strong homomorphisms, and $\phi : A \longrightarrow B$ and $\phi' : B \longrightarrow C$ are two mappings such that the equalities $\psi(\mathcal{F}(\varepsilon)) = \mathcal{K}(\phi(\varepsilon))$ and $\psi'(\mathcal{K}(\epsilon)) = \mathcal{N}(\phi'(\epsilon))$ hold for all $\varepsilon \in Supp(\mathcal{F}, A)$, $\epsilon \in Supp(\mathcal{K}, B)$. So, We can easily deduce that $\psi' \circ \psi : \mathcal{H} \longrightarrow \mathcal{H}''$ is also strong homomorphism and $\phi' \circ \phi : A \longrightarrow C$ is a mapping so that the equality

$$(\psi' \circ \psi)(\mathcal{F}(\varepsilon)) = \psi'(\psi(\mathcal{F}(\varepsilon))) = \psi'(\mathcal{K}(\phi(\varepsilon))) = \mathcal{N}(\phi'(\phi(\varepsilon))) = \mathcal{N}((\phi' \circ \phi)(\varepsilon))$$

holds for all $\varepsilon \in Supp(\mathcal{F}, A)$. Also, $(\psi' \circ \psi)_\varepsilon : (\mathcal{F}(\varepsilon), \tau_{\mathcal{F}(\varepsilon)}) \longrightarrow (\mathcal{N}((\phi' \circ \phi)(\varepsilon)), \tau''_{\mathcal{N}((\phi' \circ \phi)(\varepsilon))})$ continuous and open for all $\varepsilon \in Supp(\mathcal{F}, A)$. Hence, it is concluded that $(\psi' \circ \psi, \phi' \circ \phi) : (\mathcal{F}, A, \tau) \longrightarrow (\mathcal{N}, C, \tau'')$ is a soft topological homomorphism. □

3.2. *Soft Topological Subhyperrings*

In this subsection, we define the notion of soft topological subhyperrings and establısh some its important characterizations.

Definition 3.15. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{R}$. Then, $(\mathcal{K}, B, \tau)$ is called a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if the following conditions are satisfied:*
i. *$B \subseteq A$;*
ii. *$\mathcal{K}(b)$ is a subhyperring of $\mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$;*
iii. *The hyperoperations $+, \cdot, / : \mathcal{F}(\varepsilon) \times \mathcal{F}(\varepsilon) \longrightarrow P^*(\mathcal{F}(\varepsilon))$ are continuous with respect to the topologies induced by $\tau \times \tau$ and τ^* for all $b \in Supp(\mathcal{K}, B)$.*

Example 3.16. *Consider a soft topological hyperring $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$ and $B \subseteq A$. Then, we can easily deduce that $(\mathcal{F}|_B, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Theorem 3.17. *If $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ and $(\mathcal{N}, C, \tau)$ is a soft topological subhyperring of $(\mathcal{K}, B, \tau)$, then $(\mathcal{N}, C, \tau)$ is the soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Proof. Straightforward. □

Theorem 3.18. *Let $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ be two soft topological hyperrings over $\mathcal{R}$. Then $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$.*

Proof. Assume $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ are two soft topological hyperrings over $\mathcal{R}$. Clearly, if $(\mathcal{K}, B)$ is a soft subset of $(\mathcal{F}, A)$, it follows that $B \subseteq A$ and $\mathcal{K}(b) \subseteq \mathcal{F}(b)$ for all $b \in Supp(\mathcal{K}, B)$. Thus, $\mathcal{K}(b)$ is a topological subhyperring of $\mathcal{F}(b)$ with respect to the topology induced by τ. Thus, $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$. □

After that, we discuss some generalized properties of soft topological subhyperrings.

Theorem 3.19. *Let $(\mathcal{F}, A, \tau)$ be a soft topological hyperring over $\mathcal{H}$ and $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological subhyperrings of $(\mathcal{F}, A, \tau)$.*
i. *The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $\tilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) \neq \emptyset$.*
ii. *The extended intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$.*
iii. *The extended union of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ with the topology τ if $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha, \beta \in I$, $\alpha \neq \beta$.*

Proof. We only prove **i**., and the proofs of **ii**. and **iii**. are similar. The restricted intersection of the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ with $\bigcap_{\alpha\in I} A_\alpha \neq \emptyset$ defined by the soft set $\tilde{\bigcap}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau) = (\mathcal{F}, A, \tau)$ such that $\mathcal{F}(\varepsilon) = \bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ for all $\varepsilon \in A$. Let $\varepsilon \in Supp(\mathcal{F}, A)$. Assume $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ such that $\mathcal{F}_\alpha(\varepsilon) \neq \emptyset$ for all $\alpha \in I$. Since $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a non-empty family of soft topological subhyperrings of $(\mathcal{F}, A, \tau)$, therefore $A_\alpha \subseteq A$ and $\mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{F}(\varepsilon)$ with respect to the topology induced by τ for all $\alpha \in I$. So $\bigcap_{\alpha\in I} A_\alpha \subseteq A$ and $\bigcap_{\alpha\in I} \mathcal{F}_\alpha(\varepsilon)$ is a topological subhyperring of $\mathcal{F}(\varepsilon)$. Consequently, the family $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$
□

Besides, we can obtain the following result:

Theorem 3.20. *Let $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ be a non-empty family of soft topological hyperrings over $\mathcal{H}$ and let $(\mathcal{K}_\alpha, B_\alpha, \tau)$ be a soft topological subhyperring of $(\mathcal{F}_\alpha, A_\alpha, \tau)$ for all $\alpha \in I$. Then, $\wedge-$intersection $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{K}_\alpha, B_\alpha, \tau)$ is a soft topological subhyperring of $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ if it is non-null.*

Proof. Suppose that $\{(\mathcal{F}_\alpha, A_\alpha, \tau) \mid \alpha \in I\}$ is a non-empty family of soft topological hyperrings over $\mathcal{R}$. By Theorem 3.5 (ii), it is clear that $\widetilde{\bigvee}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$ is a soft topological hyperring over $\mathcal{R}$. Choose $\epsilon_\alpha \in Supp(\mathcal{K}_\alpha, B_\alpha)$. Then $\bigcap_{\alpha\in I} \mathcal{K}_\alpha(\epsilon_\alpha) \neq \emptyset$ which implies that $\mathcal{K}_\alpha(\epsilon_\alpha) \neq \emptyset$ for all $\alpha \in I$ and $(\epsilon_\alpha)_{\alpha\in I} \in B_i$. Further, $B_\alpha \subseteq A_\alpha$ and $\mathcal{K}_\alpha(\epsilon_\alpha)$ is a topological subhyperring of $\mathcal{F}_\alpha(\epsilon_\alpha)$ with respect to the topology induced by τ for all $\alpha \in I$ such that $\bigcap_{\alpha\in I} B_\alpha \subseteq \bigcap_{\alpha\in I} A_\alpha$ and $\bigvee_{\alpha\in I}(\mathcal{K}_\alpha(\epsilon_\alpha))$ is also a a topological subhyperring of $\bigvee_{\alpha\in I}(\mathcal{F}_\alpha(\epsilon_\alpha))$. Therefore, $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{K}_\alpha, B_\alpha, \tau)$ is a soft topological subhyperring of $\widetilde{\bigwedge}_{\alpha\in I}(\mathcal{F}_\alpha, A_\alpha, \tau)$.
□

Theorem 3.21. *Let $(\mathcal{K}, B, \tau)$ be a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. Then, the restricted intersection of $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ if it is non-null.*

Proof. Suppose that $(\mathcal{K}, B, \tau)$ is a soft topological subhypergroup of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. If it is non-null, we have that $B \subseteq A$ and $\mathcal{K}(\epsilon)$ is a topological subhyperring of $\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Thus, we can obtain easily that $A \cap B \subseteq A$ and $\mathcal{K}(\epsilon) \cap \mathcal{F}(\epsilon)$ is a topological subhyperring of $\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Hence, the restricted intersection $(\mathcal{F}, A, \tau)\tilde{\cap}(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.
□

Theorem 3.22. *Let $f : \mathcal{R} \longrightarrow \mathcal{R}'$ be a good homomorphism of the topological hyperrings $(\mathcal{F}, A, \tau')$ and $(\mathcal{K}, B, \tau')$ over $\mathcal{H}'$. Then $(f^{-1}(\mathcal{K}), B, \tau)$ is a soft topological subhyperring of $(f^{-1}(\mathcal{F}), A, \tau)$ if $(\mathcal{K}, B, \tau')$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau')$.*

Proof. Consider $(\mathcal{K}, B, \tau')$ as a soft topological subhyperring of $(\mathcal{F}, A, \tau')$ over $\mathcal{R}$. Let $\epsilon \in Supp(f^{-1}(\mathcal{K}), B)$. Because $(\mathcal{K}, B, \tau')$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau')$, we have that $B \subseteq A$ and $(\mathcal{K}(b))$ is a topological subhyperring of $(\mathcal{F}(\epsilon)$ with respect to the topology induced by τ' for all $\epsilon \in Supp(f^{-1}(\mathcal{K}), B)$. Further, since $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good topological homomorphism, so $f^{-1}(\mathcal{F})(\epsilon) = f^{-1}(\mathcal{F}(\epsilon))$ is a topological subhyperring of $f^{-1}(\mathcal{K})(\epsilon) = f^{-1}(\mathcal{K}(\epsilon))$ with respect to the topology induced by τ for all $\epsilon \in Supp(f(\mathcal{K}), B)$. Therefore, $(f^{-1}(\mathcal{K}), B, \tau)$ is a soft topological subhyperring of $(f^{-1}(\mathcal{F}), A, , \tau)$. □

Theorem 3.23. *Let $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good homomorphism of the topological hyperrings $(\mathcal{F}, A, \tau)$ and $(\mathcal{K}, B, \tau)$ over $\mathcal{R}$. Then $(f(\mathcal{K}), B, \tau')$ is a soft topological subhyperring of $(f(\mathcal{F}), A, \tau')$ over $\mathcal{H}'$ if $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$.*

Proof. Assume $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$ over $\mathcal{R}$. If $(\mathcal{K}, B, \tau)$ is a soft topological subhyperring of $(\mathcal{F}, A, \tau)$, this means that $B \subseteq A$ and $(\mathcal{K}(\epsilon))$ is a topological subhyperring of $(\mathcal{F}(\epsilon)$ with respect to the topology induced by τ for all $\epsilon \in Supp(\mathcal{K}, B)$. Also, because $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a good topological homomorphism, we have that $f(\mathcal{F})(\epsilon) = f(\mathcal{F}(\epsilon))$ is a topological subhyperring of $f(\mathcal{K})(\epsilon) = f(\mathcal{K}(\epsilon))$ with respect to the topology induced by τ' for all $\epsilon \in Supp(f(\mathcal{K}), B)$. Hence, $(f(\mathcal{K}), B, \tau')$ is a soft topological subhyperring of $(f(\mathcal{F}), A, \tau')$. □

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