**TURKISH JOURNAL OF SCIENCE**
**VOLUME 6, ISSUE 1, 36–44**
**ISSN: 2587–0971**

http://dergipark.gov.tr/tjos

# The Geometry of Ribbons

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**Abstract.** In this study, the representations of the ribbons in the 3-dimensional Euclidean space as the developable ruled surface are given. By calculating the average curvature of the ribbon surface, the results regarding the mean curvature according to the character of the centerline are obtained. In addition, examples supporting these results are given.

## 1. Introduction

In classical differential geometry, the ruled surfaces with additional property (constant mean curvature, constant Gauss curvature, minimal, etc.) are probably the simplest surface having the specified properties see for example [7],[8], [11] and references there in. These surfaces have many applications in surface modeling [12] and parametric design [13]. The structures formed are in the form of ribbon that increase in width and length in the [2] self-shaping process. The rulings don't correspond to the ribbon's central lines, but only to geometric lines that are constantly evolving during the deformation [5]. Helical ribbons are a significant class of 2-dimensional structures that often occurs in engineering and biology [14]. The predictive model for the mechanics and morphology of the stability of spiral bands is a new and important tool for research and design in various technologies such as biological sensing, nano-engineering coils for visual electronics [10].

This agreement aims to develop a common framework for discussing the above mathematical model of ribbons by classifying according to the centerline curves $\gamma(s)$. This article is structured as follows: Section 2 contains some notations and basic equations of the differential geometry of spatial curves in $\mathbb{E}^3$. Section 3 describes some geometrical properties of ruled surfaces at $\mathbb{E}^3$. Section 4 contains original results about ribbon surfaces in $\mathbb{E}^3$. These components also provide some of the key characteristics of ribbon surfaces and the construction of their curvatures. In section 5, we present some examples of ribbon surfaces. Finally, in section 6, we discuss our findings and decide details for future work.

## 2. Preliminaries

We will now analyze some notations of the differential distribution of spatial curves in 3-dimensional Euclidean space $\mathbb{E}^3$. Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a unit speed curve with $\gamma'(s) \neq 0$, where $\gamma'(s) = \frac{d\gamma}{ds}$. $T(s) = \gamma'(s)$

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Received: 5 March 2021; Accepted: 3 April 2021; Published: 30 April 2021
*Keywords*. Ruled surface, Surface reconstruction, Helical Ribbons, Serret-Frenet frame
2010 *Mathematics Subject Classification*. 53C40, 53C42
*Cited this article as:* Arslan K, Bulca B, Öztürk G. The Geometry of Ribbons. Turkish Journal of Science. 2021, 6(1), 36-44.

is a unit tangent vector and is perpendicular to $T'(s) = \gamma''(s)$. If $\gamma''(s) \neq 0$, these vectors extend on the plane of oscillation $\gamma$ in $s$. Specify the curvature of $\gamma$ with $\kappa(s) = \left\|\gamma''(s)\right\|$. If $\kappa(s) \neq 0$, then the principal principal unit of the unit $N(s)$ of the curve $\gamma$ in $s$ is given by $T'(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector $\gamma$ in $s$. Hence the Serret-Frenet formulas of $\gamma$ are

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned} \tag{1}$$

where $\tau(s)$ is the torsion of the curve $\gamma$ at $s$ [6].

A curve $\gamma = \gamma(s) : I \to \mathbb{E}^3$ with $\kappa(s) \neq 0$ is called a *conical geodesic (resp. cylindrical helix)* if the ratio $(\frac{\tau}{\kappa})'(s)$ (resp. $\frac{\tau}{\kappa}(s)$) is constant function [7]. If $\kappa(s) \neq 0$, $\tau(s)$ are both constant, then $\gamma$ is known as *circular helix (W-curve)* [6].

In [4], B.Y. Chen defined a new type of curve in three-dimensional Euclidean space called a *rectifying curve*. According to his definition, a unit speed curve $\gamma : I \to \mathbb{E}^3$ is called the rectifying curve if $\gamma$ satisfies the equation

$$\gamma(s) = m_1(s)T(s) + m_2(s)B(s) \tag{2}$$

for some real valued functions $m_1(s)$ and $m_2(s)$ [4]. By differentiating (2) and using the Frenet formulas one can obtain $m_1(s) = 1$, $m_2'(s) = 0$, $m_1\kappa - m_2\tau = 0$. As a result of these conditions, it is easy to show that the curve is a rectifying curve if and only if $\frac{\tau}{\kappa}(s) = as + b$ holds. Consequently, each rectifying curve is a kind of conical geodesic.

The Frenet motion formulas can be expounded as "if with the time variable s the motion point crosses the curve, the moving frame $\{T(s), N(s), B(s)\}$ moves based on (1). Consequently, the instantaneous rotational speed given by the Darboux vector

$$W(s) = \tau(s)T(s) + \kappa(s)B(s). \tag{3}$$

The Darboux vector is in its instantaneous axis direction of rotation and its length is

$$\omega(s) = \|W(s)\| = \sqrt{\kappa^2(s) + \tau^2(s)}.$$

The *modified Darboux* vector along the curve $\gamma$ is defined by (see [7])

$$\widetilde{W}(s) = \frac{\tau(s)}{\kappa(s)}T(s) + B(s). \tag{4}$$

## 3. Material and Method

We now deal with the ruled surfaces in Euclidean space $\mathbb{E}^3$.

**Definition 3.1** *With the ruled patch*

$$\varphi = \varphi_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s) \tag{5}$$

*the surface is called a ruled surface, where, $\gamma$ is the base curve and $\beta$ is the director of the surface. The rulings of the surface are the lines $u \longmapsto \gamma(s) + u\beta(s)$ (see [1], [7],[8], [9] and [11]).*

Let $S$ be a ruled surface. In this case, the $T_pS$ space is spanned by the following vectors;

$$\begin{aligned} \varphi_s(s,u) &= \frac{\partial\varphi_{(\gamma,\beta)}}{\partial s} = \gamma'(s) + u\beta'(s), \\ \varphi_u(s,u) &= \frac{\partial\varphi_{(\gamma,\beta)}}{\partial u} = \beta(s). \end{aligned}$$

The coefficients of the $1^{st}$ fundamental form are

$$\begin{aligned} g_{11} &= <\varphi_s(s,u), \varphi_s(s,u)>= \left\|\gamma'(s) + u\beta'(s)\right\|^2, \\ g_{12} &= <\varphi_s(s,u), \varphi_u(s,u)>= \langle\gamma'(s), \beta(s)\rangle + u\langle\beta'(s), \beta(s)\rangle, \\ g_{22} &= <\varphi_u(s,u), \varphi_u(s,u)>= \langle\beta(s), \beta(s)\rangle, \end{aligned} \tag{6}$$

where $\langle,\rangle$ is the inner product of $\mathbb{E}^3$. If the area element

$$\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\| = \sqrt{g_{11}g_{22} - g_{12}^2}, \tag{7}$$

does not vanish then $\varphi_{(\gamma,\beta)}$ is called *regular*. From now on we assume that $\varphi_{(\gamma,\beta)}$ is a regular patch. Then, the unit normal vector is

$$U(s,u) = \frac{\varphi_s(s,u) \times \varphi_u(s,u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}. \tag{8}$$

Also, the partial derivatives of second order are:

$$\begin{aligned} \varphi_{ss}(s,u) &= \gamma''(s) + u\beta''(s), \\ \varphi_{su}(s,u) &= \beta'(s), \\ \varphi_{uu}(s,u) &= 0. \end{aligned} \tag{9}$$

Using (8) with (9) the coefficients of the $2^{nd}$ fundamental form become

$$\begin{aligned} L_{11} &= <\varphi_{ss}(s,u), U>= \frac{\det(\varphi_{ss}, \varphi_s, \varphi_u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}, \\ L_{12} &= <\varphi_{su}(s,u), U>= \frac{\det(\varphi_{su}, \varphi_s, \varphi_u)}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|}, \\ L_{22} &= <\varphi_{uu}(s,u), U>= 0. \end{aligned} \tag{10}$$

Summing up these equations, one can write that the Gaussian curvature of $S$ at point $(s,u)$ is

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} = -\frac{\langle\beta'(s), \gamma'(s) \times \beta(s)\rangle^2}{\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|^2}, \tag{11}$$

and the mean curvature of $S$ is

$$\begin{aligned} H &= \frac{L_{11}g_{22} + L_{22}g_{11} - 2g_{12}L_{12}}{2\left(g_{11}g_{22} - g_{12}^2\right)} \\ &= \frac{\langle\beta,\beta\rangle \det(\varphi_{ss}, \varphi_s, \varphi_u) - 2\left(\langle\gamma',\beta\rangle + u\langle\beta',\beta\rangle\right) + \det(\gamma', \varphi_s, \varphi_u)}{2\left\|\varphi_s(s,u) \times \varphi_u(s,u)\right\|^3}. \end{aligned} \tag{12}$$

If $K$ vanishes then the ruled surface is called *developable*. Further, if $\beta'(s) \times \beta(s) = 0$, then $S$ is called cylindrical, otherwise it is non-cylindrical. The surface of $S \subset \mathbb{E}^3$ is minimal if and only if its mean curvature vanishes identically.

In [7], authors studied the *rectifying developable surfaces* given with the parametrization

$$\varphi_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s) \tag{13}$$

where $\widetilde{W}(s)$ is the *modified Darboux* vector field defined by (4). They have proved that if the rectifying developable surface given with the parametrization (13) of is a cylindrical surface (resp. conical surface) then the base curve $\gamma$ is a cylindrical helix (resp. conical helix).

## 4. Results

In [7], S. Izumiya and N. Takeuchi studied with ruled surface using the base curve $\widetilde{W}(s)$. They called them rectifying developable surfaces. In this section, we present an application of rectifying developable surface to ribbons in $\mathbb{E}^3$. However, a characterization of the mean curvature of the strip surfaces according to the character of the central line of the ribbon is given.

**Definition 4.1** *A ribbon is a rectifying developable surface defined by the ruled patch*

$$\widetilde{\varphi} = \widetilde{\varphi}_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s),\ s \in [0, L],\ u \in [-b, b] \tag{14}$$

*where, $\widetilde{W}$ is the modified Darboux vector field defined by (4) (see, [3]).*

Let $R$ be a ribbon surface given with the ruled patch (14) then the tangent space of $R$ is spanned by

$$\begin{aligned} \widetilde{\varphi}_s(s,u) &= (1 + u\rho'(s))\, T(s) \\ \widetilde{\varphi}_u(s,u) &= \rho(s)T(s) + B(s). \end{aligned} \tag{15}$$

where $\rho(s) = \frac{\tau(s)}{\kappa(s)}$ is the *harmonic curvature function of* $\gamma$. Then the coefficients of the $1^{st}$ fundamental form of $R$ are found as

$$\begin{aligned} \widetilde{g}_{11} &= <\widetilde{\varphi}_s(s,u), \widetilde{\varphi}_s(s,u)>= (1 + u\rho'(s))^2 \\ \widetilde{g}_{12} &= <\widetilde{\varphi}_s(s,u), \widetilde{\varphi}_u(s,u)>= \rho(s)\,(1 + u\rho'(s)) \\ \widetilde{g}_{22} &= <\widetilde{\varphi}_u(s,u), \widetilde{\varphi}_u(s,u)>= 1 + (\rho(s))^2. \end{aligned} \tag{16}$$

Consequently, the area element of the ribbon becomes

$$\sqrt{\widetilde{g}_{11}\widetilde{g}_{22} - \widetilde{g}_{12}^2} = \left\|\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)\right\| = \left|1 + u\rho'(s)\right| \tag{17}$$

where

$$\begin{aligned} \widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u) &= (1 + u\rho'(s))\, T(s) \times (\rho(s)T(s) + B(s)) \\ &= -(1 + u\rho'(s))\, N(s) \end{aligned} \tag{18}$$

is the surface normal. So, the unit normal vector field of $R$ becomes

$$\widetilde{U}(s,u) = \frac{\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)}{\left\|\widetilde{\varphi}_s(s,u) \times \widetilde{\varphi}_u(s,u)\right\|} = -N(s). \tag{19}$$

The second partial derivatives of $\widetilde{\varphi}(u,v)$ are expressed as follows

$$\begin{aligned} \widetilde{\varphi}_{ss}(s,u) &= u\rho''(s)T(s) + \kappa(s)\,(1 + u\rho'(s))\, N(s), \\ \widetilde{\varphi}_{su}(s,u) &= \rho'(s)T(s), \\ \widetilde{\varphi}_{uu}(s,u) &= 0. \end{aligned} \tag{20}$$

Using (19) with (20) the coefficients of the $2^{nd}$ fundamental form become

$$\begin{aligned} \widetilde{L}_{11} &= <\widetilde{\varphi}_{ss}(s,u), U>= -\kappa(s)\,(1 + u\rho'(s))\,, \\ \widetilde{L}_{12} &= <\widetilde{\varphi}_{su}(s,u), U>= 0, \\ \widetilde{L}_{22} &= <\widetilde{\varphi}_{uu}(s,u), U>= 0. \end{aligned} \tag{21}$$

From the equations (21) and (11) it can be easily seen that the ribbon $R$ is a flat surface. Furthermore, summing up (16)-(20) and using (12) we obtain the following results;

**Theorem 4.2** *Let R be a ribbon surface given by (14), then the mean curvature vector of R becomes*

$$\widetilde{H}(s,u) = -\frac{\kappa(s)\left(1+(\rho(s))^2\right)}{2\left|1+u\rho'(s)\right|} \tag{22}$$

*where* $\rho(s) = \frac{\tau(s)}{\kappa(s)}$ *is the harmonic curvature (function) of the central line* $\gamma$.

With the equation (22) we have the following results;

**Corollary 4.3** *The ribbon surface R given by the parametrization (14) can not be minimal.*

**Corollary 4.4** *Let R be a ribbon surface given by (14). If the center line of R is a circular helix then the mean curvature of the ribbon is constant i.e., the ribbon is of constant mean curvature.*

**Corollary 4.5** *Let R be a ribbon surface given by (14). If the center line of R is a cylindrical helix then the mean curvature of the ribbon turns into* $\widetilde{H}(s,u) = \delta\kappa(s)$, *where* $\delta = -\frac{1+\rho}{2}$ *is a constant function.*

The following result contains a characterization of the Serret-Frenet curvatures of the spherical (constant slope) helix curves.

**Lemma 4.6** *[6] Let* $\gamma = \gamma(s)$ *be a unit-speed curve in* $\mathbb{E}^3$ *that has constant slope* $cot\theta = \frac{\tau}{\kappa}$ *with respect to a unit vector* $\overrightarrow{u} \in \mathbb{E}^3$, *where* $0 < \theta < \frac{\pi}{2}$. *Assume also that* $\gamma$ *lies on a sphere of radius* $r > 0$ *then the curvature and torsion of* $\gamma$ *are given by*

$$\kappa^2(s) = \frac{1}{r^2 - s^2\tan^2\theta} = \frac{1}{r^2 - s^2\cot^2\theta}, \tau^2(s) = \frac{1}{r^2\tan^2\theta - s^2}. \tag{23}$$

**Proposition 4.7** *Let R be a ribbon surface whose center line is a spherical helix given with the Serret-Frenet curvatures* $\kappa$ *and* $\tau$. *Then, the mean curvature H of the ribbon R is given by* $\widetilde{H} = \lambda\kappa(s)$,*where* $\lambda$ *is a constant function defined by* $\lambda = -\frac{1+\cot^2\theta}{2}$

*Proof.* Assume that the centerline of the strip is a spherical slope curve (helix) then the ratio of curvatures must be constant. Thus, with the help of equations (23) and (22) we get the result. □

**Corollary 4.8** *Let R be a ribbon surface whose center line is a conical geodesic, i.e.* $\rho''(s) = 0$. *Then, the mean curvature* $\widetilde{H}$ *of the ribbon R is the multiple of the curvature* $\kappa(s)$,*with a smooth function*

$$\mu(s,u) = \frac{1+(as+b)^2}{2\left|1+au\right|},\ a,b \in \mathbb{R}. \tag{24}$$

The *geodesic curvature*, the *normal curvature* and the *geodesic torsion* of the surface associated the curve $\gamma(s)$ are defined as follows;

$$\kappa_g = \langle U \times T, T' \rangle, \kappa_n = \langle U, \gamma'' \rangle, \tau_g = \langle U \times U', T' \rangle \tag{25}$$

where $U$ is the unit normal of the surface. From this consideration, a curve $\gamma(s)$ is an *asymptotic line* (resp. *geodesic line* or *principal line*) if and only if normal curvature $\kappa_n$ (resp. geodesic curvature $\kappa_g$ or geodesic torsion $\tau_g$) vanishes identically [6].

**Proposition 4.9** *The center line* $\gamma$ *of the ribbon R is geodesically principal and the normal curvature of R corresponds to the curvature* $\kappa$ *of* $\gamma$.

*Proof.* By the use of Serret-Frenet frame (1) with (25) the geodesic curvature, the normal curvature and the geodesic torsion of $R$ become

$$\begin{aligned} \kappa_g &= \langle U \times T, T' \rangle = \kappa \langle B, N \rangle = 0, \\ \tau_g &= \langle U \times U', T' \rangle = \kappa \langle D, N \rangle = 0, \\ \kappa_n &= \langle U, \gamma'' \rangle = -\kappa \langle N, N \rangle = -\kappa, \end{aligned} \tag{26}$$

respectively. □

**Definition 4.10** *A unit speed planar curve $\gamma : I \longrightarrow \mathbb{E}^2$ whose curvature is a given piecewise-continuous function $\kappa : I \longrightarrow \mathbb{R}^+$ is parametrized by*

$$\gamma(s) = \left( \int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b \right); \ \theta(s) = \int \kappa(s)ds + c \tag{27}$$

*where, $a, b, c$ are constant of integration [6].*

If the center line of the ribbon is a regular curve $\gamma(s) = (x(s), y(s), 0)$ then the the resultant ribbon $R$ becomes a cylindrical ruled surface with the parametrization

$$\widetilde{\varphi}_{(\gamma,\widetilde{W})}(s,u) = \left( \int \cos\theta(s)ds + a, \int \sin\theta(s)ds + b, u \right), s \in [0, L], u \in [-b, b]. \tag{28}$$

Thus, we have the following result;

**Corollary 4.11** *Let $R$ be a ribbon surface given by the parametrization (28). Then the mean curvature of $R$ is a multiple of the curvature $\kappa$ of the form $\widetilde{H} = -\frac{\kappa(s)}{2}$.*

## 5. Visualization

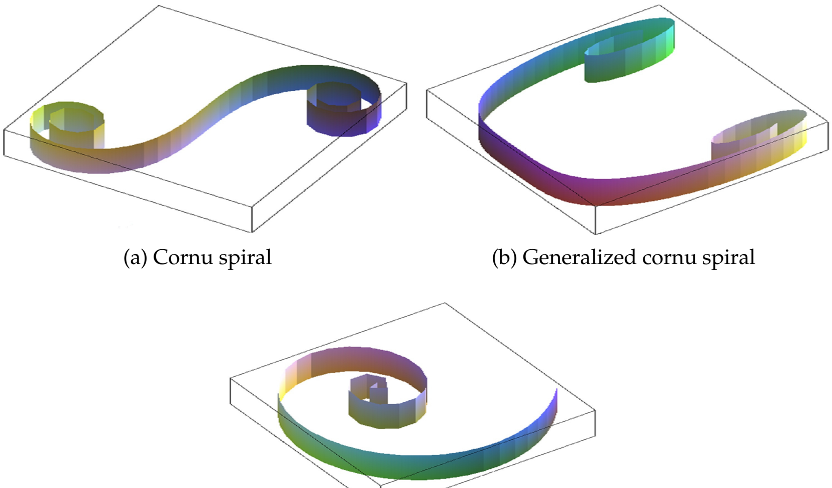
Geometric models of curves and surfaces have an important place in computer-aided geometric design. Therefore, in the present section, geometric visualization of some ribbon models is given with the help of maple.

**Example 5.1** *In this example we construct three kind of ribbon using the plane curve given with the parametrization (21);*

*(a) For $\kappa(s) = as + b$ the center line is a Cornu spiral and the graph of resultant ribbon is cylindrical (Figure 1-(a)).*

*(b) For $\kappa(s) = as^2 + bs + c$ the center line is a generalized Cornu spiral and the graph of resultant ribbon is cylindrical (Figure 1-(b)).*

*(c) For $\kappa(s) = \frac{a}{s+b}$ the center line is a logarithmic spiral and the graph of resultant ribbon is cylindrical (Figure 1-(c)).*



(a) Cornu spiral (b) Generalized cornu spiral

(c) Logarithmic spiral

Figure 1: Ribbon Surfaces in $\mathbb{E}^3$

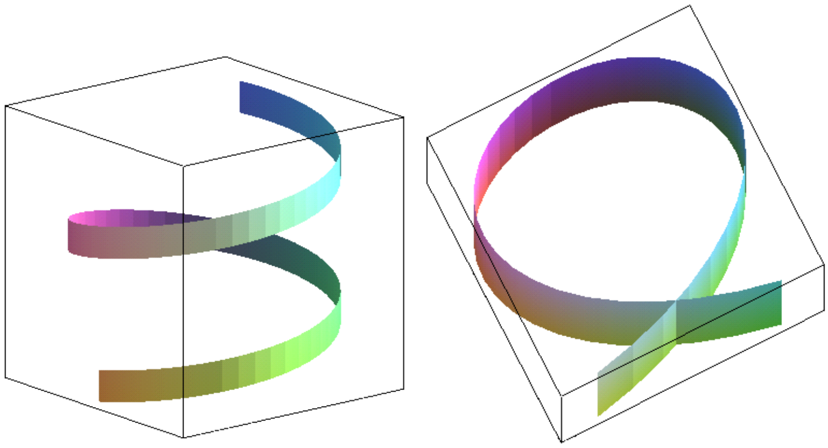
**Example 5.2** *We take the center line curve $\gamma$ as a right circular helix*

$$\gamma(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin(\frac{s}{c}), \frac{bs}{c}\right),\ a^2 + b^2 = c^2.$$

*The Serret-Frenet curvatures of $\gamma$ are constant functions $\kappa(s) = \frac{a}{c^2}, \tau(s) = \frac{b}{c^2}$ and $\cot\theta(s) = \frac{b}{a}$ . A simple calculation shows that the ribbon $R$ has constant mean curvature $H = -\frac{1}{2a}$. In Figure 2-(a) we pictured the ribbon taking the values $a = 3$, $b = 4$ and $c = 5$.*

**Example 5.3** *Consider the planar curve $\gamma(s) = (s\cos(s), s\sin s, 0)$ with curvature $\kappa(s) = -\frac{2+s^2}{(1+s^2)^{3/2}}$. The resultant surface becomes a braid ribbon which has self intersection (Figure 2-(b)). Further, the mean curvature of the ribbon becomes*

$$\widetilde{H}(s,u) = -\frac{2+s^2}{2(1+s^2)^{3/2}}.$$



(a) Cylindrical ribbon (b) The braid ribbon

Figure 2: Ribbon Surfaces in $\mathbb{E}^3$

## 6. Conclusions

In conclusion, the paper presents a simple method for constructing developable surface patches bounded by space curves. These surfaces have many applications in surface modeling and parametric design. The most relevant feature of this construction is that the parametrization of the resultant ruled surface gives a ribbon in 3-dimensional Euclidean space. The method is founded on finding a special type of ruled surface taking the director curve as Darboux vector of the base curve. It has been shown that the mean curvature of the helical ribbon surfaces are related with the Serret-Frenet curvature of the center line of the ribbon. Nowadays, helical ribbons getting popular in nanoengineering. The detailed exploration of this analogy is the subject of future work. Especially, We would also like to extend our calculations to a parallel transport frame of ribbon configurations.

## References


[1] Alegre P, Arslan K, Carriazo A, Murathan C and Öztürk G. Some Special Types of Developable Ruled Surface. Hacettepe Journal of Mathematics and Statics. 39(3), 2010, 319-325.

[2] Armon S, Aharoni H, Moshe M and Sharon E. Shape selection in chiral ribbons: from seed pods to supramolecular assemblie. Soft Matter. 10, 2014, 2733-2740.

[3] Bohr J and Markvorsen S. Ribbon Crystals. Plos One. 8(10), 2013, 1-7.

[4] Chen BY. When does the position vector of a space curve always lie in its rectifying plane?. The American Mathematical Monthly. 110(2), 2003, 147-152.

[5] Charrondière R, Descoubes F-B, Neukirch S, Romero V. Numerical modeling of inextensible elastic ribbons with curvature-based elements. Computer Methods in Applied Mechanics and Engineering. 364, 2020, 1-24.

[6] Gray A. Modern Differential Geometry of Curves and Surfaces. Boca Raton. FL, CRC Press, 1993.

[7] Izumiya S and Takeuchi N. New special curves and developable surfaces. Turkish Journal of Mathematics. 28(2), 2004, 153-164.

[8] Izumiya S and Takeuchi N. Special curves and ruled surfaces. Contributions to Algebra and Geometry. 44(1), 2003, 200-212.

[9] Izumiya S, Katsumi H and Yamashi T. The rectifying developable and spherical Darboux image of a space curve. Geometry and Topology of Caustics'98. 50, 1999, 137-149.

[10] Kong XY and Wang ZL. Spontaneous Polarization-Induced Nanohelixes, Nanosprings, and Nanorings of Piezoelectric Nanobelts. Nano Letters. 3(12), 2003, 1625-1631.

[11] Özyılmaz E and YaylıY. On the closed space-like developable ruled surface. Hadronic Journal. 23(4), 2000, 439-456.

[12] Peternell M, Pottmann H and Ravani B. On the computational geometry of ruled surfaces. Computer Aided Design. 31(1), 1999, 17-32.

[13] Sánchez MIG, Uriel AG and Rios IG. Ruled Surfaces and Parametric Design. Congreso Internacional de Expresión Gráfica Arquitectónica EGA 2018, Spain, 2018, 231-241.

[14] Srivastava S, Santos A, Critchley K, Kim K-S, Podsiadlo P, Sun K, Lee J, Xu C, Lilly GD, Glotzer SC, Kotov NA. Light-Controlled Self-Assembly of Semiconductor Nanoparticles into Twisted Ribbons. Science. 327(5971), 2010, 1355-1359.