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Fourth Derivative Block Method for Solving Two-point Singular Boundary Value Problems and Related Stiff Problems

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Abstract. This paper contains the formulation of an algorithm for solving two-point singular nonlinear boundary value problems of ordinary differential equations. This method is basically a fourth derivative block method obtained from the collocation and interpolation of an assumed derivatives and functional of a basis function. Its implementation was on the evaluation of derivatives of the given smooth first derivative function $u'(t)$ up to the fourth derivative, at some points t. It is proved that the algorithm is consistent, zero-stable and convergent. Errors for uniform step lengths are also investigated and presented. Numerical examples are provided to show the efficiency of the algorithm.

1. Introduction

Considering the following singular non-linear two-point boundary value problem

$$a(t)u''(t) + b(t)u'(t) = f(t,u,u'), \quad t \in [0,1], \; u'(0) = 0, \; u(1) = u_b \tag{1}$$

with assumption that

$$a(0) = 0, \; a(t) > 0, \; t \in (0,1), b(0) \neq 0, f(0,u(0),u'(0)) = 0 \tag{2}$$

with coefficients $a(t)$ and $b(t)$ are differentiable functions on $[0,1]$ and $f(t,u,u')$ is assumed continuous on $\omega := [0,1] \times \Re^2$. It could be observed that the problem is singular at the initial point $t = 0$. If $a(t)$ and $b(t)$ satisfy

$$a(1) = 0, b(1) \neq 0 \text{and} f(1,u(1),u'(1)) = 0, \tag{3}$$

it is also singular at point $t = 1$.

Problems of the form (1) satisfying conditions (2) to (3) posses property which make the solutions difficult to obtain or the numerical solutions are poor, and as such special techniques are required for their effective solution. Problems (1) with condition (2) are one-point singular in nature while (1) with conditions (2) and (3) are two-point singular problems. These singular two-point problems happen much of the time in

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numerous models, for example, electro-hydrodynamics and some warm blasts, and in the recent times, have been researched by utilizing some numerical techniques by Baxley ([3] and [4]), Qu and Agarwal [5], Chawla and Subramanian ([6] and Chawla, Subramanian and Sathi [7]). The numerical approach explored in these literature include cubic and quintic spline methods, and the collocation methods. An even-order two-point boundary value problem solutions were obtained by Liu[8]. A continuous generic algorithm was used by Arqub, Abo-Hammour, Momani and Shawegfeh [9] for solving the form of problem in (1). A subdivision collocation method for solving two point boundary value problems of order three was considered by Mustafa and Ejaz [10]. Parametric difference method was used by Pandey [11] in the solution of two-point boundary value problem. An alternative approach was considered by Ghomanjani and Shateyi [12] using the Bezoer curve method with an orthogonal based Bernstein polynomials constructed by the Gram-Schmidt technique. Solutions of one-point singular Lane-endem equations and related stiff problems were effectively solved using some new numerical techniques by Ogunniran, Haruna & Adeniyi [13] and also Ogunniran [14] obtained a class of multi-derivative method for solution of some singular Advection equations of partial differential equations. An extensive linear analysis was carried out on some Runge-kutta methods on their possibilities in the solution of one point singular Lane-Endem equations by Ogunniran, Tayo, Haruna and Adebisi [15]. Extensive analysis for the possibility of existence and uniqueness of solution for a two-point boundary value problems for ordinary differential equations was carried out by Eloe and Henderson [16] in their paper titled; two-point boundary value problems for ordinary differential equations, uniqueness implies existence. However, two-point boundary value problems may exist in problems of order greater than two as found in Agarwal and Kelevedjiev [17]. This paper presents a unique approach on the solution of fourth-order two-point boundary value problems.

2. Method

Recently, lots of attention has been on obtaining more effective and proficient methods for solving stiff problems and subsequently a wide class of methods have been proposed. A possibly decent numerical method for solving stiff systems of ordinary differential equations need to have good accuracy and some reasonably wide region of absolute stability (Dahlquist, 1963). According to Hairer and Wanner (1996), the search for high order A-stable multi-step methods is carried out in two main ways: using high derivatives of solutions and including some additional stages, such as off-step points or super-future points. And this transforms into the many field of general multi-step methods.
Throughout the formulation of this method, except where stated otherwise, the transformation

$$T = \frac{2(t - t_n)}{kh} - 1, \tag{4}$$

where $k = 3$ is the step number and h is the step length, a small distance taken that does not entirely leave the interval.
For purpose of obtaining an approximation for (1), we assume a continuous approximation for $u_n(t)$ of a three step fourth derivative method of the form:

$$u(x) \approx \alpha(t)u_n + \sum_{i=1}^{4} h^i \beta_i(t) f_n^{(i-1)} + \sum_{i=1}^{4}\sum_{j=1}^{3} h^i \gamma_{ij}(t) f_{(n+j)}^{(i-1)} \tag{5}$$

for $u' = f(t,u)$ where $f(t,u)$ is continuous and differentiable, u_n is an approximation to $u(t_n)$, $t_n = nh; h > 0$ and $f_m^{(j)} = f^{(j)}(t_m, u_m)$ such that:

$$f^{(0)}(t_m, u_m) \tag{6}$$

$$f^{(j)}(t_m, u_m) = \frac{\partial f^{(j-1)}(t,u)}{\partial t} + f(t,u)\frac{\partial f^{(j-1)}(t,u)}{\partial u} \tag{7}$$

To this end, approximation of the exact solution $u(t)$ was sought by evaluating the function:

$$u(t) = \sum_{j=0}^{16} a_j t^j \tag{8}$$

where $a_j,\ j = 0(1)16$ are coefficients determined, t^j are the basis functions of degree 16.
While ensuring that (5) corresponds with the analytical solution at the end point t_n, the following conditions were imposed on $u(x)$ and its derivatives; $u^{(k)}(t),\ \ k = 1(1)4$

$$\begin{gathered} u(t_{n+j}) = u_{n+j},\ j = 0 \\ u'(t_{n+j}) = f_{n+j},\ j = 0,1,2,3. \\ u''(t_{n+j}) = g_{n+j},\ j = 0,1,2,3. \\ u'''(t_{n+j}) = h_{n+j},\ j = 0,1,2,3. \\ u^{(iv)}(t_{n+j}) = i_{n+j},\ j = 0,1,2,3. \end{gathered} \tag{9}$$

while the conditions of (9) are imposed on (8), the following system equations were obtained;

$$\begin{gathered} a_0 = y_n \\ a_1 = f_n \\ a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 \\ +9a_9 + 10a_{10} + 11a_{11} + 12a_{12} + 13a_{13} + 14a_{14} + 15a_{15} + 16a_{16} = f_{n+1} \\ a_1 + 4a_2 + 12a_3 + 32a_4 + 80a_5 + 192a_6 + 448a_7 + 1024a_8 \\ +2304a_9 + 5120a_{10} + 11264a_{11} + 24576a_{12} + 53248a_{13} \\ +114688a_{14} + 245760a_{15} + 524288a_{16} = f_{n+2} \\ a_1 + 6a_2 + 27a_3 + 108a_4 + 405a_5 + 1458a_6 + 5103a_7 + 17496a_8 \\ +59049a_9 + 196830a_{10} + 649539a_{11} + 2125764a_{12} + 6908733a_{13} \\ +22320522a_{14} + 71744535a_{15} + 229582512a_{16} = f_{n+3} \\ 2a_2 = g_n \\ 2a_2 + 6a_3 + 12a_4 + 20a_5 + 30a_6 + 42a_7 + 56a_8 + 72a_9 \\ +90a_{10} + 110a_{11} + 132a_{12} + 156a_{13} + 182a_{14} + 210a_{15} + 240a_{16} = g_{n+1} \\ 2a_2 + 12a_3 + 48a_4 + 160a_5 + 480a_6 + 1344a_7 + 3584a_8 \\ +9216a_9 + 23040a_{10} + 56320a_{11} + 135168a_{12} + 319488a_{13} + 745472a_{14} \\ +1720320a_{15} + 3932160a_{16} = g_{n+2} \\ 2a_2 + 18a_3 + 108a_4 + 540a_5 + 2430a_6 + 10206a_7 + 40824a_8 \\ +157464a_9 + 590490a_{10} + 2165130a_{11} + 7794468a_{12} + 27634932a_{13} \\ +96722262a_{14} + 334807830a_{15} + 1147912560a_{16} = g_{n+3} \\ 6a_3 = h_n \\ 6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \\ +720a_{10} + 990a_{11} + 1320a_{12} + 1716a_{13} + 2184a_{14} \\ +2730a_{15} + 3360a_{16} = h_{n+1} \\ 6a_3 + 48a_4 + 240a_5 + 960a_6 + 3360a_7 + 10752a_8 + 32256a_9 \\ +92160a_{10} + 253440a_{11} + 675840a_{12} + 1757184a_{13} + 4472832a_{14} \\ +11182080a_{15} + 27525120a_{16} = h_{n+2} \\ 6a_3 + 72a_4 + 540a_5 + 3240a_6 + 17010a_7 + 81648a_8 + 367416a_9 \\ +1574640a_{10} + 6495390a_{11} + 25981560a_{12} + 101328084a_{13} \\ +386889048a_{14} + 1450833930a_{15} + 5356925280a_{16} = h_{n+3} \\ 24a_4 = i_n \\ 24a_4 + 120a_5 + 360a_6 + 840a_7 + 1680a_8 + 3024a_9 \\ +5040a_{10} + 7920a_{11} + 11880a_{12} + 17160a_{13} + 24024a_{14} \\ +32760a_{15} + 43680a_{16} = i_{n+1} \\ 24a_4 + 240a_5 + 1440a_6 + 6720a_7 + 26880a_8 + 96768a_9 \\ +322560a_{10} + 1013760a_{11} + 3041280a_{12} + 8785920a_{13} + 24600576a_{14} \\ +67092480a_{15} + 178913280a_{16} = i_{n+2} \end{gathered}$$

$$
\begin{gathered}
24\,a_4 + 360\,a_5 + 3240\,a_6 + 22680\,a_7 + 136080\,a_8 + 734832\,a_9 \\
+3674160\,a_{10} + 17321040\,a_{11} + 77944680\,a_{12} + 337760280\,a_{13} + 1418593176\,a_{14} \\
+5803335720\,a_{15} + 23213342880\,a_{16} = i_{n+3}
\end{gathered} \tag{10}
$$

Solving (10), $a_j,\ \ j = 0(1)16$ were obtained, the values were substituted in (8) and related term were collected in $u_n, f_n, f_{n+1}, f_{n+2}, f_{n+3}, g_n, g_{n+1}, g_{n+2}, g_{n+3}, h_n, h_{n+1}, h_{n+2}, h_{n+3}, i_n, i_{n+1}, i_{n+2}, i_{n+3}$ to obtain:

$$
\begin{gathered}
u(t) = \alpha_n u_n + h\beta_1(t) f_n + h^2\beta_2(t) g_n + h^3\beta_3(t) h_n + h^4\beta_4(t) i_n + h\left[\gamma_{11}(t) f_{n+1} + \gamma_{12}(t) f_{n+2} + \gamma_{13}(t) f_{n+3}\right] \\
+h^2\left[\gamma_{21}(t) g_{n+1} + \gamma_{22}(t) g_{n+2} + \gamma_{23}(t) g_{n+3}\right] + h^3\left[\gamma_{31}(t) h_{n+1} + \gamma_{32}(t) h_{n+2} + \gamma_{33}(t) h_{n+3}\right] \\
+h^4\left[\gamma_{41}(t) i_{n+1} + \gamma_{42}(t) i_{n+2} + \gamma_{43}(t) i_{n+3}\right]
\end{gathered} \tag{11}
$$

where

$$
\begin{gathered}
\alpha_n(t) = 1 \\
\beta_1(t) = t - \frac{54391\,t^5}{1296} + \frac{713717\,t^6}{3888} - \frac{62588555\,t^7}{163296} + \frac{276696055\,t^8}{559872} - \frac{1259675\,t^9}{2916} + \frac{9332263\,t^{10}}{34992} - \frac{5049247\,t^{11}}{42768} \\
+\frac{5227135\,t^{12}}{139968} - \frac{1251365\,t^{13}}{151632} + \frac{18545\,t^{14}}{15309} - \frac{2477\,t^{15}}{23328} + \frac{4711\,t^{16}}{1119744} \\
\beta_2(t) = -81\,t^5 + \frac{2007\,t^6}{4} - \frac{19035\,t^7}{14} + \frac{68955\,t^8}{32} - \frac{320945\,t^9}{144} + \frac{101285\,t^{10}}{64} \\
-\frac{138705\,t^{11}}{176} + \frac{35265\,t^{12}}{128} - \frac{13795\,t^{13}}{208} + \frac{4695\,t^{14}}{448} - \frac{47\,t^{15}}{48} + \frac{21\,t^{16}}{512} \\
\beta_3(t) = \frac{1863\,t^5}{16} - \frac{10359\,t^6}{16} + \frac{367875\,t^7}{224} - \frac{637125\,t^8}{256} + \frac{89695\,t^9}{36} - \frac{27605\,t^{10}}{16} + \frac{148197\,t^{11}}{176} \\
-\frac{18525\,t^{12}}{64} + \frac{14285\,t^{13}}{208} - \frac{75\,t^{14}}{7} + \frac{95\,t^{15}}{96} - \frac{21\,t^{16}}{512} \\
\beta_4(t) = \frac{529\,t^5}{81} - \frac{36821\,t^6}{972} + \frac{1026995\,t^7}{10206} - \frac{11217545\,t^8}{69984} + \frac{1974065\,t^9}{11664} - \frac{17350807\,t^{10}}{139968} \\
+\frac{2742691\,t^{11}}{42768} - \frac{6550475\,t^{12}}{279936} + \frac{894155\,t^{13}}{151632} - \frac{957245\,t^{14}}{979776} + \frac{1117\,t^{15}}{11664} - \frac{4711\,t^{16}}{1119744} \\
\gamma_{11}(t) = 1/2\,t^2 - \frac{4711\,t^5}{270} + \frac{61909\,t^6}{864} - \frac{72497\,t^7}{504} + \frac{33704285\,t^8}{186624} - \frac{338717\,t^9}{2187} + \frac{549763\,t^{10}}{5832} - \\
\frac{147109\,t^{11}}{3564} + \frac{603953\,t^{12}}{46656} - \frac{5983\,t^{13}}{2106} + \frac{67715\,t^{14}}{163296} - \frac{211\,t^{15}}{5832} + \frac{533\,t^{16}}{373248} \\
\gamma_{12}(t) = -\frac{891\,t^5}{10} + \frac{1881\,t^6}{4} - \frac{32103\,t^7}{28} + \frac{13485\,t^8}{8} - \frac{474731\,t^9}{288} + \frac{71729\,t^{10}}{64} - \frac{189809\,t^{11}}{352} \\
+\frac{23465\,t^{12}}{128} - \frac{17941\,t^{13}}{416} + \frac{2995\,t^{14}}{448} - \frac{59\,t^{15}}{96} + \frac{13\,t^{16}}{512} \\
\gamma_{13}(t) = -\frac{2511\,t^5}{40} + \frac{11277\,t^6}{32} - \frac{101169\,t^7}{112} + \frac{354405\,t^8}{256} - \frac{25249\,t^9}{18} + \frac{3935\,t^{10}}{4} - \frac{42811\,t^{11}}{88} \\
+\frac{10849\,t^{12}}{64} - \frac{4241\,t^{13}}{104} + \frac{1445\,t^{14}}{224} - \frac{29\,t^{15}}{48} + \frac{13\,t^{16}}{512} \\
\gamma_{21}(t) = -\frac{193\,x^5}{90} + \frac{4037\,x^6}{324} - \frac{32233\,x^7}{972} + \frac{617465\,x^8}{11664} - \frac{3921371\,x^9}{69984} + \frac{1920017\,x^{10}}{46656} \\
-\frac{202945\,x^{11}}{9504} + \frac{729457\,x^{12}}{93312} - \frac{199861\,x^{13}}{101088} + \frac{107395\,x^{14}}{326592} - \frac{755\,x^{15}}{23328} + \frac{533\,x^{16}}{373248} \\
\gamma_{22}(t) = 1/6\,t^3 - \frac{533\,t^5}{180} + \frac{7015\,t^6}{648} - \frac{123413\,t^7}{6048} + \frac{766211\,t^8}{31104} - \frac{119969\,t^9}{5832} + \frac{119401\,t^{10}}{9720} - \\
\frac{25199\,t^{11}}{4752} + \frac{12791\,t^{12}}{7776} - \frac{335\,t^{13}}{936} + \frac{353\,t^{14}}{6804} - \frac{35\,t^{15}}{7776} + \frac{11\,t^{16}}{62208} \\
\gamma_{23}(t) = -\frac{81\,t^5}{10} + \frac{99\,t^6}{2} - \frac{3753\,t^7}{28} + \frac{6801\,t^8}{32} - \frac{63241\,t^9}{288} + \frac{6219\,t^{10}}{40} - \frac{27139\,t^{11}}{352} \\
+\frac{5149\,t^{12}}{192} - \frac{2671\,t^{13}}{416} + \frac{113\,t^{14}}{112} - \frac{3\,t^{15}}{32} + \frac{t^{16}}{256}\gamma_{31}(t) = \frac{81\,t^5}{8} - \frac{909\,t^6}{16} + \frac{32589\,t^7}{224} - \frac{28497\,t^8}{128} + \frac{16205\,t^9}{72} - \frac{3147\,t^{10}}{20} + \frac{13639\,t^{11}}{176} \\
-\frac{2579\,t^{12}}{96} + \frac{167\,t^{13}}{26} - \frac{113\,t^{14}}{112} + \frac{3\,t^{15}}{32} - \frac{t^{16}}{256} \\
\gamma_{32}(t) = \frac{23\,t^5}{90} - \frac{241\,t^6}{162} + \frac{8999\,t^7}{2268} - \frac{49373\,t^8}{7776} + \frac{157207\,t^9}{23328} - \frac{6032\,t^{10}}{1215} + \frac{8189\,t^{11}}{3168} \\
-\frac{14773\,t^{12}}{15552} + \frac{8129\,t^{13}}{33696} - \frac{1097\,t^{14}}{27216} + \frac{31\,t^{15}}{7776} - \frac{11\,t^{16}}{62208} \\
\gamma_{33}(t) = 1/24\,t^4 - \frac{11\,t^5}{45} + \frac{145\,t^6}{216} - \frac{365\,t^7}{324} + \frac{79441\,t^8}{62208} - \frac{1495\,t^9}{1458} + \frac{11591\,t^{10}}{19440} - \frac{25\,t^{11}}{99} \\
+\frac{1201\,t^{12}}{15552} - \frac{35\,t^{13}}{2106} + \frac{65\,t^{14}}{27216} - \frac{t^{15}}{4860} + \frac{t^{16}}{124416} \\
\gamma_{41}(t) = -\frac{27\,t^5}{10} + \frac{57\,t^6}{4} - \frac{963\,t^7}{28} + \frac{199\,t^8}{4} - \frac{41233\,t^9}{864} + \frac{30527\,t^{10}}{960} - \frac{5279\,t^{11}}{352} \\
+\frac{1921\,t^{12}}{384} - \frac{37\,t^{13}}{32} + \frac{79\,t^{14}}{448} - \frac{23\,t^{15}}{1440} + \frac{t^{16}}{1536} \\
\gamma_{42}(t) = -\frac{27\,t^5}{20} + \frac{123\,t^6}{16} - \frac{2241\,t^7}{112} + \frac{7979\,t^8}{256} - \frac{3469\,t^9}{108} + \frac{5501\,t^{10}}{240} - \frac{1015\,t^{11}}{88} \\
+\frac{785\,t^{12}}{192} - t^{13} + \frac{9\,t^{14}}{56} - \frac{11\,t^{15}}{720} + \frac{t^{16}}{1536} \\
\gamma_{43}(t) = -\frac{t^5}{90} + \frac{7\,t^6}{108} - \frac{131\,t^7}{756} + \frac{1081\,t^8}{3888} - \frac{6905\,t^9}{23328} + \frac{3403\,t^{10}}{15552} - \frac{1087\,t^{11}}{9504} \\
+\frac{1313\,t^{12}}{31104} - \frac{121\,t^{13}}{11232} + \frac{197\,t^{14}}{108864} - \frac{7\,t^{15}}{38880} + \frac{t^{16}}{124416}
\end{gathered} \tag{12}
$$

Evaluating (12) at t_{n+1}, t_{n+2} and t_{n+3} yield the desired discrete block method below:

$$
\begin{array}{c}
u_{n+1} = -\frac{34637\, i_{n+3}}{1868106240}h^4 - \frac{2617\, i_{n+2}}{1064448}h^4 - \frac{37603\, i_{n+1}}{5322240}h^4 + \frac{50857\, i_{n}}{373621248}h^4 \\
+\frac{401183\, h_{n+3}}{934053120}h^3 + \frac{224473\, h_{n+2}}{11531520}h^3 + \frac{965\, h_{n+1}}{2306304}h^3 + \frac{4135199\, h_{n}}{934053120}h^3 \\
-\frac{20331329\, g_{n+3}}{5604318720}h^2 - \frac{398083\, g_{n+2}}{3294720}h^2 - \frac{7097579\, g_{n+1}}{23063040}h^2 + \frac{331133249\, g_{n}}{5604318720}h^2 \\
+h\frac{12457877\, f_{n+3}}{1120863744} + h\frac{1071529\, f_{n+2}}{4612608} + h\frac{1730473\, f_{n+1}}{4612608} + h\frac{427519381\, f_{n}}{1120863744} + u_n \\
u_{n+2} = -h^4\frac{16\, i_{n+3}}{729729} - h^4\frac{i_{n+2}}{297} - h^4\frac{64\, i_{n+1}}{10395} + h^4\frac{509\, i_{n}}{3648645} \\
+h^3\frac{1864\, h_{n+3}}{3648645} + h^3\frac{206\, h_{n+2}}{6435} + h^3\frac{584\, h_{n+1}}{45045} + h^3\frac{470\, h_{n}}{104247} \\
-h^2\frac{47504\, g_{n+3}}{10945935} - h^2\frac{10441\, g_{n+2}}{45045} - h^2\frac{8864\, g_{n+1}}{45045} + h^2\frac{654539\, g_{n}}{10945935} \\
+h\frac{29272\, f_{n+3}}{2189187} + h\frac{6577\, f_{n+2}}{9009} + h\frac{7864\, f_{n+1}}{9009} + h\frac{839939\, f_{n}}{2189187} + u_n \\
u_{n+3} = -h^4\frac{81\, i_{n+3}}{512512} + h^4\frac{729\, i_{n+2}}{197120} - h^4\frac{729\, i_{n+1}}{197120} + h^4\frac{81\, i_{n}}{512512} \\
+h^3\frac{6327\, h_{n+3}}{1281280} + h^3\frac{41553\, h_{n+2}}{1281280} + h^3\frac{41553\, h_{n+1}}{1281280} + h^3\frac{6327\, h_{n}}{1281280} \\
-h^2\frac{162531\, g_{n+3}}{2562560} + h^2\frac{194643\, g_{n+2}}{2562560} - h^2\frac{194643\, g_{n+1}}{2562560} + h^2\frac{162531\, g_{n}}{2562560} \\
+h\frac{28905\, f_{n+3}}{73216} + h\frac{80919\, f_{n+2}}{73216} + h\frac{80919\, f_{n+1}}{73216} + h\frac{28905\, f_{n}}{73216} + u_n
\end{array}
\tag{13}
$$

2.1. Order and Error Constant

Applying Taylor's series expansion on (5) and collecting like terms, we have the difference equation

$$l[u(t);h] = c_0u(t) + c_1hy^{(1)}(t) + c_2hy^{(2)}(t) + \cdots + c_qhy^{(q)}(t) + \cdots \tag{14}$$

where

$$
\left.
\begin{array}{c}
c_0 = 1 - \alpha_n \\
c_1 = 3 - \beta_i - \sum_{j=1}^{3}\gamma_{j3} \\
\vdots \\
c_q = \frac{3^q}{q!} - \frac{3^{q-1}}{(q-1)!}\sum_{j=1}^{3}\gamma_{j3}
\end{array}
\right\}
\tag{15}
$$

According to Henrici (1962), a method has order p if

$$l[u(t);h] = o(h^{p+1}) \tag{16}$$

where

$$c_0 = c_1 = \cdots = c_p = 0 \text{ but } c_{p+1} \neq 0. \tag{17}$$

Using this principle, the order and error constant of (13) are shown below

Evaluating Point	Order	Error Constant
t_{n+1}	16	$\frac{183}{4883993354240000}$
t_{n+2}	16	$\frac{1231}{62585703900720000}$
t_{n+3}	16	$\frac{570391}{3204388039716864000}$

The method (13) is consistent since order of the method is 16 which is greater than 1.

2.2. Zero-stability

This relates to a phenomenon where the step size $h \to 0$. Taking limit of (13) as $h \to 0$, we have:

$$u_{n+1} = u_{n+2} = u_{n+3} = u_n \tag{18}$$

which can be written in matrix form as

$$IU_i - B_0U_{i-1} = 0 \tag{19}$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_i = \begin{pmatrix} u_{n+1} \\ u_{n+2} \\ u_{n+3} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{i-1} = \begin{pmatrix} u_i \\ u_i \\ u_i \end{pmatrix}$$

According to Lambert (1975), a block method is zero-stable if the roots r_k of the first characteristic polynomial $\xi(r) = det|Ir - B_0|$ does not exceed 1 i.e. $|r_k| \leq 1$. The first characteristic polynomial of method (13) is given by

$$r^2(r-1) = 0 \tag{20}$$

The roots of (20) are $r = 0, 0, 1$ in which all is < 1, thus Method (13) is zero-stable.

2.3. *Convergence*

According to Henrici (1962), we can establish the convergence of the block method since consistency and zero-stability are necessary and sufficient reasons for convergence.

2.4. *Linear Stability*

Practically, the robustness of a method is reliably found with $h > 0$, this implies that the convergence of a method is a necessary but not a sufficient condition for a method to be useful. Linear stability is a conceptional behaviour of numerical methods concerned with the behaviour of the method when $h > 0$ and its region of absolute stability. This is a concept different from zero-stability. The linear stability properties of the derived method is determined by expressing it in a form applicable to the test problem:

$$u' = \lambda u, \quad \text{for which} \;\; u'_n = \lambda u_n, \; u''_n = \lambda^2 u_n, \cdots, u_n^{(n)} = \lambda^n u_n., \quad \lambda < 0 \tag{21}$$

to yield:

$$U_{\mu+1} = M(z) U_\mu, \quad z = h\lambda \tag{22}$$

where the amplification matrix $M(z)$ is given by:

$$M(z) = (A^{(0)} - zB^{(0)} - z^2C^{(0)} - z^3D^{(0)} - z^4E^{(0)})^{-1}(A^{(1)} + zB^{(1)} + z^2C^{(1)} + z^3D^{(1)} + z^4E^{(1)}). \tag{23}$$

where $z = h\lambda$.

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} \frac{1730473}{4612608} & \frac{1071529}{4612608} & \frac{12457877}{1120863744} \\ \frac{7864}{9009} & \frac{6577}{9009} & \frac{29272}{2189187} \\ \frac{80919}{73216} & \frac{80919}{73216} & \frac{28905}{73216} \end{pmatrix} \quad B^{(1)} = \begin{pmatrix} 0 & 0 & \frac{427519381}{1120863744} \\ 0 & 0 & \frac{839939}{2189187} \\ 0 & 0 & \frac{28905}{73216} \end{pmatrix}$$

$$C^{(0)} = \begin{pmatrix} -\frac{7097579}{23063040} & -\frac{398083}{3294720} & -\frac{20331329}{5604318720} \\ -\frac{8864}{45045} & -\frac{10441}{45045} & -\frac{47504}{10945935} \\ -\frac{194643}{2562560} & \frac{194643}{2562560} & -\frac{162531}{2562560} \end{pmatrix} \quad C^{(1)} = \begin{pmatrix} 0 & 0 & \frac{331133249}{5604318720} \\ 0 & 0 & \frac{654539}{10945935} \\ 0 & 0 & \frac{162531}{2562560} \end{pmatrix}$$

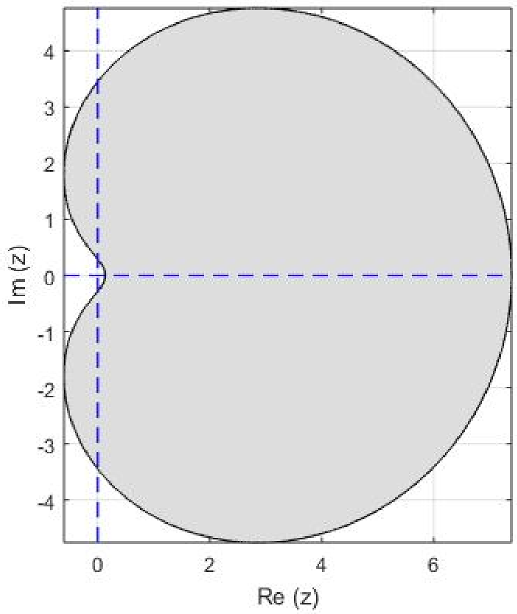

Figure 1: Region of Absolute Stability of Method (13)

$$D^{(0)} = \begin{pmatrix} \frac{965}{2306304} & \frac{224473}{11531520} & \frac{401183}{934053120} \\ \frac{584}{45045} & \frac{206}{6435} & \frac{1864}{3648645} \\ \frac{41553}{1281280} & \frac{41553}{1281280} & \frac{6327}{1281280} \end{pmatrix} \quad D^{(1)} = \begin{pmatrix} 0 & 0 & \frac{4135199}{934053120} \\ 0 & 0 & \frac{470}{104247} \\ 0 & 0 & \frac{6327}{1281280} \end{pmatrix}$$

$$E^{(0)} = \begin{pmatrix} -\frac{37603}{5322240} & -\frac{2617}{1064448} & -\frac{34637}{1868106240} \\ -\frac{64}{10395} & -\frac{1}{297} & -\frac{16}{729729} \\ -\frac{729}{197120} & \frac{729}{197120} & -\frac{81}{512512} \end{pmatrix} \quad E^{(1)} = \begin{pmatrix} 0 & 0 & \frac{50857}{373621248} \\ 0 & 0 & \frac{509}{3648645} \\ 0 & 0 & \frac{81}{512512} \end{pmatrix}$$

The matrix $M(z)$ has eigenvalues $\xi_1, \xi_2, \cdots, \xi_m = 0, 0, \cdots, \xi_m$, where the dominant eigenvalue ξ_m is the stability function $R(z)$ which is a rational function with real coefficients, m is the order of $R(z)$,

$$R(z) = \frac{\begin{matrix} 28561\,z^{12} + 1164410\,z^{11} + 25844325\,z^{10} + 401535225\,z^{9} + 4765597305\,z^{8} + 44819838000\,z^{7} \\ +338397658200\,z^{6} + 2046767184000\,z^{5} + 9765253436400\,z^{4} + 35606883312000\,z^{3} \\ +93666717144000\,z^{2} + 158855192496000\,z + 130821923232000 \end{matrix}}{\begin{matrix} z^{12} - 170\,z^{11} + 14325\,z^{10} - 754425\,z^{9} + 26611305\,z^{8} - 648043200\,z^{7} \\ +11234575800\,z^{6} - 141313788000\,z^{5} + 1292613260400\,z^{4} - 8445396960000\,z^{3} \\ +37600178616000\,z^{2} - 102788653968000\,z + 130821923232000 \end{matrix}} \tag{24}$$

The stability function and plot for the method is as given below:

3. Numerical Examples

This section contains some two-point singular boundary value problems, their conditions and true solutions as found in literature.

Test Problem 1 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = \beta t^{\beta-2}(1+\beta+\beta t^{\beta}) \\ t \in (0,1), u'(0) = 0, u(1) = e \\ u(t) = e^{t^{\beta}} \end{array}\right\} \quad (25)$$

Test Problem 2 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = 3cos(t) - tsin(t) \\ t \in (0,1), u'(0) = 0, u(1) = cos1 + sin1 \\ u(t) = cost + tsint. \end{array}\right\} \quad (26)$$

Test Problem 3 [5]

$$\left.\begin{array}{c} u''(t) + \frac{2}{t}u'(t) = -2(e^{u} + e^{\frac{u}{2}}) \\ t \in (0,1), u'(0) = 0, u(1) = 0 \\ u(t) = 2log\dfrac{2}{1+t^2} \end{array}\right\} \quad (27)$$

Test Problem 4 [11]

The boundary value problem below arose from the analysis of the confinement of a plasma column by radiation pressure with different boundary conditions,

$$\left.\begin{array}{c} u''(t) = \lambda sinh(\lambda u(t)), \ \ 0 < t < 1 \\ \text{subject to boundary conditions} \\ u'(0) = 1, \text{ and } u(1) = 0. \\ u(t) = sinh(t) \end{array}\right\} \quad (28)$$

4. Discussion of Results and Conclusion

The following formula

$$\lim_{maxt} |u(t) - u_i(t)| \quad (29)$$

where $u(t)$ is the exact solution and $u_i(t)$ is the numerical solution evaluated at some $t \in [0,1]$, was used in the computation of maximum errors. Numerical methods were programmed on Windows 10 operating system in MATLAB 9.2 environment on 8.00GB RAM HP Pavilion x360 Convertible, 64-bits Operating System, x64-based processor Intel(R) Core(TM) i3-7100U CPU @ 2.40GHz.
The following table display the comparison of performances for new method against existing methods with Computational time.

Table 1: Table of Comparison of Maximum Errors with Existing Methods Using Different h, $\beta = 1$ for Test Problem 1

Test Problem	Method	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
1	Qu & Agarwal (1997)	1.720000 E(-2)	2.800000 E(-3)	2.660000 E(-4)
	Method (13)	5.714330 E(-5)	5.714330 E(-5)	5.716299 E(-5)
2	Qu & Agarwal (1997)	1.090000 E(-5)	1.080000 E(-6)	7.890000 E(-8)
	Method (13)	5.403023 E(-9)	5.403023 E(-9)	5.461926 E(-9)
3	Qu & Agarwal (1997)	1.200000 E(-3)	1.070000 E(-4)	8.030000 E(-6)
	Method (13)	2.021798 E(-8)	2.021798 E(-8)	1.200000 E(-8)

Table 2: Table of Comparison of Maximum Errors with Existing Method Using Different h and λ, for Test Problem 4

λ	Method	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$
0.1	Pandey (2018)	1.1055857 E(-1)	1.1045857 E(-1)	1.1032658 E(-1)
	Method (13)	1.387182 E(-4)	1.387225 E(-5)	1.387229 E(-8)
0.15	Pandey (2018)	1.7244926 E(-1)	1.7190818 E(-1)	1.7155264 E(-1)
	Method (13)	1.559298 E(-4)	1.559312 E(-5)	1.559399 E(-8)
0.2	Pandey (2018)	2.374965 E(-1)	2.358833 E(-1)	0.0000 E(0)
	Method (13)	0.000000 E(0)	0.000000 E(0)	0.000000 E(0)

[1)]

Table 3: Table of Computational Time of Method (13) Measured in *seconds*

Test Problem	h	Computation Time
1	$\frac{1}{8}$	0.3438
	$\frac{1}{16}$	0.4063
	$\frac{1}{32}$	0.4844
2	$\frac{1}{8}$	0.2500
	$\frac{1}{16}$	0.3250
	$\frac{1}{32}$	0.3350
3	$\frac{1}{8}$	0.2110
	$\frac{1}{16}$	0.2520
	$\frac{1}{32}$	0.3200
4	$\frac{1}{8}$	0.3250
	$\frac{1}{16}$	0.3255
	$\frac{1}{32}$	0.3525

[1)] Tables 1 and 2 show the numerical computational results on test problems considered. Extensive comparison was done with existing methods in literature. The method was demonstrated on some Examples and the superiority of the derived method over existing methods was established. It is worthy to say that the derived methods exhibit stronger computational strength executed under very low computational times as shown in Table 3. Figures 2-9 show the graphical comparison for solutions of derived method with exact and error distribution curve across the selected interval.

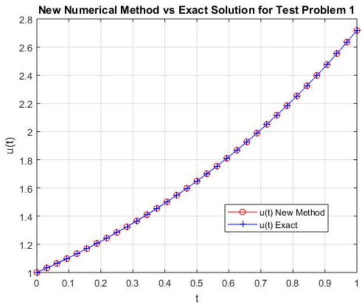

Figure 2: Method (13) vs Exact for Test Problem 1

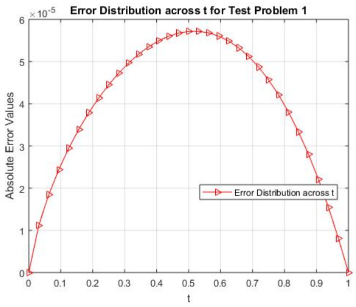

Figure 3: Error Distribution along t of Method (13) for Test Problem 1

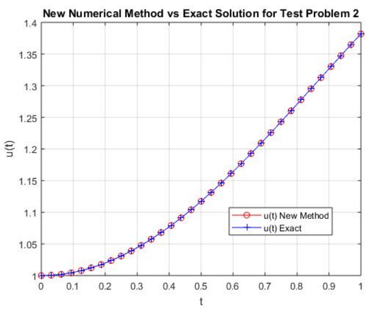

Figure 4: Method (13) vs Exact for Test Problem 2

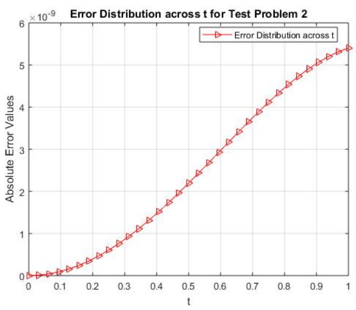

Figure 5: Error Distribution along t of Method (13) for Test Problem 2

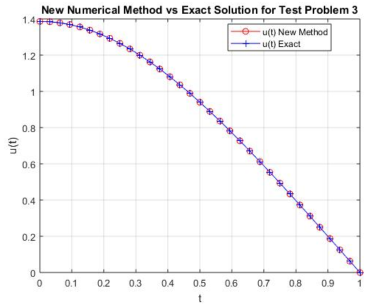

Figure 6: Method (13) vs Exact for Test Problem 3

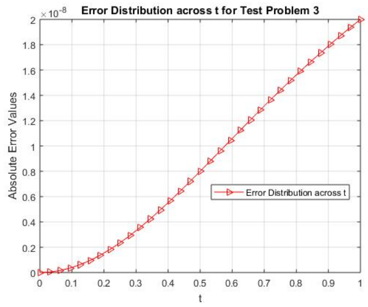

Figure 7: Error Distribution along t of Method (13) for Test Problem 3

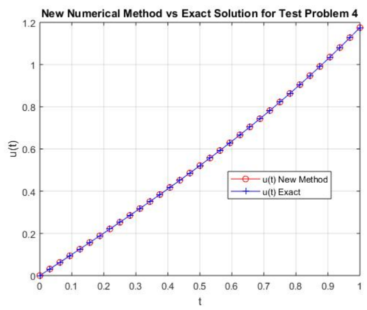

Figure 8: Method (13) vs Exact for Test Problem 4

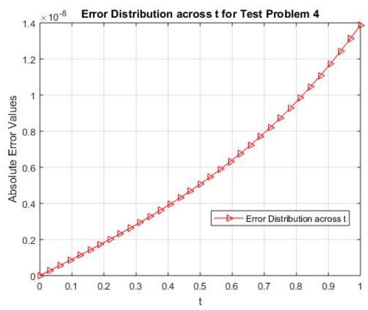

Figure 9: Error Distribution along t of Method (13) for Test Problem 4

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