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Some special Smarandache ruled surfaces by Frenet Frame in E^3 -I

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Abstract. The paper introduces some new special ruled surfaces with the base TNB- Smarandache curve where the unit vector of the generator is taken as one of other Frenet vectors and their linear combinations. The geometric properties with reference to fundamental forms such as minimality and developability of each generated surface are examined by Gauss and mean curvatures. An example is also given by considering the famous Viviani's curve.

1. Introduction

The theory of surfaces is an important branch of differential geometry. A typical surface is defined as an image of a function with two real valued variables (domain) by a mapping to 2- or 3-dimensional space. As a special type of surfaces, the ruled surfaces are defined to be one parameter family of lines. The simplest formulation makes these surfaces popular to refer for purposes on geometric modeling. Therefore, they are subjected in many areas such as engineering, architectural designs, computer graphics, automobile industry, etc [1, 2]. Since they are mostly referred in geometric designs sometimes to deal with real world problems and more frequently to model the real objects, introducing new ruled surfaces generated by different methods will lead new potentials to the related fields. Providing their characteristics may also enable easy adaptations for interested researchers. The basic theory related to ruled surfaces can be found in many differential geometry textbooks such as [3–7]. Recently, Ouarab, (2021a) put forth a method to generate new ruled surfaces in by taking the advantage of the idea of Smarandache geometry. By assigning the base curve as one of the Smarandache curves and taking the generator as the another vector element of Frenet frame, she introduced these ruled surfaces as Smarandache ruled surfaces according to Frenet frame in [8]. The same method of generating such ruled surfaces is applied to the Darboux frame by Ouarab, (2021b) in [9] and according to the alternative frame by Ouarab, (2021c) in [10]. Motivated by this, in this study, we address new ruled surfaces by considering some linear combinations of Frenet vectors as a Smarandache curve. Then, we study some characteristics of these ruled surfaces and present an example regarding to Viviani's curve to illustrate each surface.

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2. Preliminaries

We comprise the basic concepts which will be used throughout the paper in this section. Let $\alpha : I \to E^3$ be a regular unit speed curve. The very well-known Frenet apparatus is given by following identities:

$$T = \alpha', \quad N = \frac{\alpha''}{\|\alpha''\|}, \quad B = T \wedge N, \quad \kappa = \|\alpha''\|, \quad \tau = \langle N', B \rangle,$$

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N, \ [12]$$

On the other hand, a Smarandache curve of is a regular curve generated by the position vector of the following form

$$\gamma = \frac{fT + gN + hB}{\sqrt{f^2 + g^2 + h^2}} \tag{1}$$

where f, g and h are real functions. For $\forall s \in I$ the vector γ corresponds to a differentiable curve. If each f, g and h is considered to be a constant function then the curves drawn by the γ vector are known as Smarandache curves [11]. There are many studies in the literature with the context of Smarandache curves by applying different frames and considering different spaces. For more detail see [11–13].
A ruled surface, on the other hand is a one parameter family of lines and it has the following parameterization

$$X(s, v) = \alpha(s) + vr(s). \tag{2}$$

The normal vector field of the ruled surface , is given as

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{3}$$

while the Gauss and mean curvatures are defined by

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \tag{4}$$

respectively [1–5]. The coefficients appeared at (4) are known to be the coefficients of first and second fundamental forms and calculated by followings:

$$E = \langle X_s, X_s \rangle, \quad F = \langle X_s, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \tag{5}$$

$$e = \langle X_{ss}, N_X \rangle, \quad f = \langle X_{sv}, N_X \rangle, \quad g = \langle X_{vv}, N_X \rangle. \tag{6}$$

3. Some special Smarandache Ruled Surfaces according to Frenet Frame in E^3

Let us recall the relation (1) . If $f = g = h = 1$, then the corresponding curve whose position vector is $\vec{\gamma} = \dfrac{\vec{T} + \vec{N} + \vec{B}}{\sqrt{3}}$ is called as the $TNB-$ Smarandache curve. Next, let us consider the ruled surfaces whose base is $TNB-$ Smarandache curve and the genarator is the one of following unit vectors

$$\vec{T}, \quad \vec{N}, \quad \vec{B}, \quad \vec{r_1} = \frac{\vec{T} + \vec{N}}{\sqrt{2}}, \quad \vec{r_2} = \frac{\vec{T} + \vec{B}}{\sqrt{2}}, \quad \vec{r_3} = \frac{\vec{N} + \vec{B}}{\sqrt{2}}, \quad \vec{r_4} = \frac{\vec{T} + \vec{N} + \vec{B}}{\sqrt{3}}.$$

We examine the properties of these seven ruled surfaces by means of Gaussian and mean curvatures.

Definition 3.1. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T, N, B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{T}$ moves along on the $TNB-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{F}(s, v) = \frac{1}{\sqrt{3}}\left(\left(1 + \sqrt{3}v\right)T + N + B\right).$$

The first and second partial derivatives of the surface $\mathcal{F}(s,v)$ are given in respective order as follows:

$$\mathcal{F}_s = \frac{1}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau + \sqrt{3}v\kappa\right)N + \tau B\right), \quad \mathcal{F}_v = T, \quad \mathcal{F}_{sv} = \kappa N, \quad \mathcal{F}_{vv} = 0,$$
$$\mathcal{F}_{ss} = \frac{1}{\sqrt{3}}\left(\left(-\kappa' - \kappa^2 + \tau\kappa - \sqrt{3}v\kappa^2\right)T - \left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right)N + \left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right)B\right).$$

To formulate the normal vector field of $\mathcal{F}(s,v)$ denoted by $N_{\mathcal{F}}$, we first compute

$$\mathcal{F}_s \wedge \mathcal{F}_v = \frac{1}{\sqrt{3}}\left(\tau N - \left(\kappa - \tau + \sqrt{3}v\kappa\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{F}_s \wedge \mathcal{F}_v\| = \frac{1}{\sqrt{3}}\sqrt{\tau^2 + \left(\kappa - \tau + \sqrt{3}v\kappa\right)^2} = \frac{1}{\sqrt{3}}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}.$$

Hence, we obtain

$$N_{\mathcal{F}} = \frac{\tau N - \left(\kappa - \tau + \sqrt{3}v\kappa\right)B}{\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{F}} = \frac{1}{3}\left(\kappa^2 + \tau^2 + \left(\kappa - \tau + \sqrt{3}v\kappa\right)^2\right), \qquad F_{\mathcal{F}} = \frac{\kappa}{\sqrt{3}}, \qquad G_{\mathcal{F}} = 1,$$
$$e_{\mathcal{F}} = \frac{-\tau\left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right) - \left(\kappa - \tau + \sqrt{3}v\kappa\right)\left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right)}{\sqrt{3}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}},$$
$$f_{\mathcal{F}} = \frac{\kappa\tau}{\sqrt{3}\sqrt{\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2}}, \qquad g_{\mathcal{F}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{F}} = -\frac{\kappa^2\tau^2}{\left(\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2\right)^2},$$
$$H_{\mathcal{F}} = -\frac{\sqrt{3}\tau\left(\kappa^2 + \tau^2 - \kappa' + \tau' - \sqrt{3}v\kappa'\right) + \sqrt{3}\left(\kappa - \tau + \sqrt{3}v\kappa\right)\left(\tau' + \tau\kappa - \tau^2 + \sqrt{3}v\tau\kappa\right) + 2\kappa^2\tau}{2\left(\kappa^2 + 2\tau^2 - 2\kappa\tau + 2\sqrt{3}v\left(\kappa^2 - \kappa\tau\right) + 3v^2\kappa^2\right)^{\frac{3}{2}}}.$$

Corollary 3.2. *If α is a planar curve then the ruled surface $\mathcal{F}(s,v)$ is both developable and minimal.*

Definition 3.3. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{N}$ moves along on the $TNB-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{U}(s,v) = \frac{1}{\sqrt{3}}\left(T + \left(1 + \sqrt{3}v\right)N + B\right).$$

The first and second partial derivatives of the surface $\mathcal{U}(s,v)$ are given in respective order as follows:

$$\mathcal{U}_s = \frac{1}{\sqrt{3}}\left(-\kappa\left(1+\sqrt{3}v\right)T + (\kappa-\tau)N + \tau\left(1+\sqrt{3}v\right)\right), \qquad \mathcal{U}_v = N, \qquad \mathcal{U}_{sv} = -\kappa T + \tau B, \qquad \mathcal{U}_{vv} = 0,$$

$$\mathcal{U}_{ss} = \frac{1}{\sqrt{3}}\left(-\left(\kappa'\left(1+\sqrt{3}v\right) + \kappa(\kappa-\tau)\right)T + \left(\kappa' - \tau' - \left(1+\sqrt{3}v\right)\left(\kappa^2+\tau^2\right)\right)N + \left(\tau(\kappa-\tau) + \tau'\left(1+\sqrt{3}v\right)\right)B\right).$$

To formulate the normal vector field of $\mathcal{U}(s,v)$ denoted by $N_{\mathcal{U}}$, we first compute

$$\mathcal{U}_s \wedge \mathcal{U}_v = \frac{1}{\sqrt{3}}\left(\tau\left(1+\sqrt{3}v\right)T - \kappa\left(1+\sqrt{3}v\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{U}_s \wedge \mathcal{U}_v\| = \frac{1}{\sqrt{3}}\left(1+\sqrt{3}v\right)\sqrt{\kappa^2+\tau^2}.$$

Hence, we obtain

$$N_{\mathcal{U}} = \frac{\tau T - \kappa B}{\sqrt{\kappa^2+\tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{U}} = \frac{1}{3}\left(\left(\kappa^2+\tau^2\right)\left(1+\sqrt{3}v\right)^2 + (\kappa-\tau)^2\right), \qquad F_{\mathcal{U}} = \frac{(\kappa-\tau)}{\sqrt{3}}, \qquad G_{\mathcal{U}} = 1,$$

$$e_{\mathcal{U}} = -\frac{(\tau\kappa' + \kappa\tau')\left(1+\sqrt{3}v\right) + 2\tau\kappa(\kappa-\tau)}{\sqrt{3}\sqrt{\kappa^2+\tau^2}}, \qquad f_{\mathcal{U}} = -\frac{2\kappa\tau}{\sqrt{\kappa^2+\tau^2}}, \qquad g_{\mathcal{U}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{U}} = -\frac{12\kappa^2\tau^2}{\left(1+\sqrt{3}v\right)^2\left(\kappa^2+\tau^2\right)^2},$$

$$H_{\mathcal{U}} = -\frac{3(\tau\kappa' + \kappa\tau')\left(1+\sqrt{3}v\right) + 18\kappa\tau(\kappa-\tau)}{2\sqrt{3}\left(1+\sqrt{3}v\right)\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}.$$

Corollary 3.4.

- *If α is a planar curve, then the ruled surface $\mathcal{F}(s,v)$ is both developable and minimal.*
- *If α is a circular helix with equal curvatures, then the ruled surface $\mathcal{F}(s,v)$ is minimal.*

Definition 3.5. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{B}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{Z}(s,v) = \frac{1}{\sqrt{3}}\left(T + N + \left(1+\sqrt{3}v\right)B\right).$$

The first and second partial derivatives of the surface $\mathcal{Z}(s,v)$ are given in respective order as follows:

$$\mathcal{Z}_s = \frac{1}{\sqrt{3}}\left(-\kappa T + \left(\kappa - \tau - \sqrt{3}v\tau\right)N + \tau B\right), \mathcal{Z}_v = B, \qquad \mathcal{Z}_{sv} = -\tau N, \qquad \mathcal{Z}_{vv} = 0,$$

$$\mathcal{Z}_{ss} = \frac{1}{\sqrt{3}}\left(-\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)T + \left(\kappa' - \tau' - \sqrt{3}v\tau' - \kappa^2 - \tau^2\right)N + \left(\tau' + \tau\kappa - \tau^2 - \sqrt{3}v\tau^2\right)B\right).$$

To formulate the normal vector field of $\mathcal{Z}(s,v)$ denoted by $N_{\mathcal{Z}}$, we first compute

$$\mathcal{Z}_s \wedge \mathcal{Z}_v = \frac{1}{\sqrt{3}}\left(\left(\kappa - \tau - \sqrt{3}v\tau\right)T - \kappa N\right).$$

When the norm is taken, we have

$$\|\mathcal{Z}_s \wedge \mathcal{Z}_v\| = \frac{1}{\sqrt{3}}\sqrt{\left(\kappa - \tau - \sqrt{3}v\tau\right)^2 + \kappa^2} = \frac{1}{\sqrt{3}}\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}.$$

Hence, we obtain

$$N_{\mathcal{Z}} = \frac{\left(\kappa - \tau - \sqrt{3}v\tau\right)T - \kappa N}{\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{Z}} = \frac{1}{3}\left(\left(\kappa^2 + \tau^2\right)\left(\kappa - \tau - \sqrt{3}v\tau\right)^2\right), \qquad F_{\mathcal{Z}} = \frac{\tau}{\sqrt{3}}, \qquad G_{\mathcal{Z}} = 1,$$

$$e_{\mathcal{Z}} = \frac{-\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)\left(\kappa - \tau - \sqrt{3}v\tau\right) - \kappa\left(\kappa' - \tau' - \sqrt{3}v\tau' - \kappa^2 - \tau^2\right)}{\sqrt{3}\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}},$$

$$f_{\mathcal{Z}} = \frac{\kappa\tau}{\sqrt{2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2}}, \qquad g_{\mathcal{Z}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{Z}} = \frac{-3\kappa^2\tau^2}{\left(2\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2\right)^2},$$

$$H_{\mathcal{Z}} = \frac{-\sqrt{3}\left(\kappa' T + \kappa^2 - \tau\kappa - \sqrt{3}v\kappa\tau\right)\left(\kappa - \tau - \sqrt{3}v\tau\right) - \sqrt{3}\left(\kappa\kappa' - \kappa\tau' - \sqrt{3}v\kappa\tau' - \kappa^3 + \kappa\tau^2\right)}{2\left(\kappa^2 + \tau^2 - 2\kappa\tau - 2\sqrt{3}v\left(\kappa\tau - \tau^2\right) + 3v^2\tau^2.\right)^{\frac{3}{2}}}.$$

Corollary 3.6. *If α is a planar curve, then the ruled surface $\mathcal{F}(s,v)$ is developable.*

Definition 3.7. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_1}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{S}(s,v) = \frac{1}{\sqrt{3}}(T + N + B) + \frac{v}{\sqrt{2}}(T + N).$$

The first and second partial derivatives of the surface $\mathcal{S}(s,v)$ are given in respective order as follows:

$$\mathcal{S}_s = \frac{1}{\sqrt{6}}\left(-\kappa\left(\sqrt{2} + v\sqrt{3}\right)T + \left(\kappa\left(\sqrt{2} + v\sqrt{3}\right) - \sqrt{2}\tau\right)N + \tau\left(\sqrt{2} + v\sqrt{3}\right)B\right),$$

$$\mathcal{S}_v = \frac{1}{\sqrt{2}}(T + N), \quad \mathcal{S}_{sv} = \frac{1}{\sqrt{2}}(-\kappa T + \kappa N + \tau B), \quad \mathcal{S}_{vv} = 0,$$

$$\mathcal{S}_{ss} = \frac{1}{\sqrt{6}}\left\{\begin{array}{l} \left(-\sqrt{2}\left(\kappa' + \kappa^2 - \kappa\tau\right) - v\sqrt{3}\left(\kappa' + \kappa^2\right)\right)T \\ \quad + \left(\sqrt{2}\left(\kappa' - \tau' - \kappa^2 - \tau^2\right) + v\sqrt{3}\left(\kappa' - \kappa^2 - \tau^2\right)\right)N \\ \quad + \left(\sqrt{2}\left(\tau' - \tau^2 + \kappa\tau\right) + v\sqrt{3}\left(\tau' + \kappa\tau\right)\right)B \end{array}\right\}$$

To formulate the normal vector field of $\mathcal{S}(s,v)$ denoted by $N_{\mathcal{S}}$, we first compute

$$\mathcal{S}_s \wedge \mathcal{S}_v = \frac{1}{2\sqrt{6}}\left(-\tau\left(2+\sqrt{6}v\right)T + \tau\left(2+\sqrt{6}v\right)N + \left(2\tau - 2\kappa\left(2+\sqrt{6}v\right)\right)B\right).$$

When the norm is taken, we have

$$\|\mathcal{S}_s \wedge \mathcal{S}_v\| = \frac{1}{\sqrt{6}}\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}.$$

Hence, we obtain

$$N_{\mathcal{S}} = \frac{-\tau\left(2+\sqrt{6}v\right)T + \tau\left(2+\sqrt{6}v\right)N + \left(2\tau - 2\kappa\left(2+\sqrt{6}v\right)\right)B}{2\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\mathcal{S}} = \frac{1}{6}\left(4\left(\kappa^2-\kappa\tau+\tau^2\right)+2\sqrt{6}v\left(2\kappa^2-\kappa\tau+\tau^2\right)+3v^2\left(2\kappa^2+\tau^2\right)\right),$$
$$F_{\mathcal{S}} = -\frac{\tau}{\sqrt{6}}, \qquad G_{\mathcal{S}} = 1,$$

$$e_{\mathcal{S}} = \frac{\begin{array}{l}4\kappa'\tau-4\kappa\tau'-4\tau^3+\sqrt{6}\tau\tau'+\sqrt{6}\kappa\tau^2+6\kappa\tau^2-6\kappa^2\tau \\ +\sqrt{6}v\left(4\kappa'\tau-2\tau\tau'-4\kappa\tau'-3\kappa^2\tau+2\kappa\tau^2-2\kappa^2\tau-2\tau^3\right)+3v^2\left(2\tau\kappa'-2\kappa\tau'-2\kappa^2\tau-\tau^3\right)\end{array}}{2\sqrt{3}\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^{\frac{1}{2}}},$$
$$f_{\mathcal{S}} = \frac{\tau^2-2\kappa\tau-\sqrt{6}\kappa\tau v}{\sqrt{2}\sqrt{(4\kappa^2+3\tau^2-4\kappa\tau)+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)}}, \qquad g_{\mathcal{S}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{S}} = \frac{-3\left(\tau^2-2\kappa\tau-\sqrt{6}\kappa\tau v\right)^2}{\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^2},$$

$$H_{\mathcal{S}} = \frac{\begin{array}{l}4\kappa'\tau-4\kappa\tau'-6\tau^3+\sqrt{6}\tau\tau'+\left(\sqrt{6}+10\right)\kappa\tau^2-6\kappa^2\tau \\ +\sqrt{6}v\left(4\kappa'\tau-2\tau\tau'-4\kappa\tau'-3\kappa^2\tau+2\kappa\tau^2-2\kappa^2\tau-2\tau^3\right)+3v^2\left(2\tau\kappa'-2\kappa\tau'-2\kappa^2\tau-\tau^3\right)\end{array}}{\left(2/\sqrt{3}\right)\left(4\kappa^2+3\tau^2-4\kappa\tau+2\sqrt{6}v\,(2\kappa^2+\tau^2-\kappa\tau)+3v^2\,(2\kappa^2+\tau^2)\right)^{\frac{3}{2}}}.$$

Corollary 3.8. *If α is a planar curve then the ruled surface $\mathcal{S}(s,v)$ is developable.*

Definition 3.9. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_2}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{Q}(s,v) = \frac{1}{\sqrt{3}}(T+N+B)+\frac{v}{\sqrt{2}}(T+B).$$

The first and second partial derivatives of the surface $\boldsymbol{Q}(s,v)$ are given in respective order as follows:

$$\boldsymbol{Q}_s = \frac{1}{\sqrt{6}}\left(-\sqrt{2}\kappa T + (\kappa-\tau)\left(\sqrt{2}+v\sqrt{3}\right)N + \sqrt{2}\tau B\right),$$
$$\boldsymbol{Q}_v = \frac{1}{\sqrt{2}}(T+B), \quad \boldsymbol{Q}_{vv} = 0, \quad \boldsymbol{Q}_{sv} = \frac{1}{\sqrt{2}}(\kappa-\tau)N,$$
$$\boldsymbol{Q}_{ss} = \frac{1}{\sqrt{6}}\left\{\begin{array}{c} \left(\sqrt{2}\left(-\kappa' - \kappa^2 + \tau\kappa\right) + \sqrt{3}v\left(-\kappa^2+\tau\kappa\right)\right)T \\ +\left(-\sqrt{2}\left(\kappa^2+\tau^2+\tau'-\kappa'\right) - \sqrt{3}v\left(\tau'-\kappa'\right)\right)N \\ +\left(\sqrt{2}\left(\tau'+\kappa\tau-\tau^2\right) + \sqrt{3}v\left(\tau\kappa-\tau^2\right)\right)B \end{array}\right\}$$

To formulate the normal vector field of $\boldsymbol{Q}(s,v)$ denoted by $N_{\boldsymbol{Q}}$, we first compute

$$\boldsymbol{Q}_s \wedge \boldsymbol{Q}_v = \frac{1}{2\sqrt{6}}\left\{(\kappa-\tau)\left(2+\sqrt{6}v\right)T + 2(\kappa+\tau)N - (\kappa-\tau)\left(2+\sqrt{6}v\right)B\right\}.$$

When the norm is taken, we have

$$\|\boldsymbol{Q}_s \wedge \boldsymbol{Q}_v\| = \frac{1}{\sqrt{6}}\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}.$$

Hence, we obtain

$$N_{\boldsymbol{Q}} = \frac{(\kappa-\tau)\left(2+\sqrt{6}v\right)T + 2(\kappa+\tau)N - (\kappa-\tau)\left(2+\sqrt{6}v\right)B}{2\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_{\boldsymbol{Q}} = \frac{1}{6}\left(4\kappa^2+4\tau^2-4\tau\kappa+2\sqrt{6}v(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right),$$
$$F_{\boldsymbol{Q}} = \frac{\tau-\kappa}{\sqrt{6}}, \qquad G_{\boldsymbol{Q}} = 1,$$

$$e_{\boldsymbol{Q}} = \frac{4\sqrt{3}\left(\tau\kappa'-\kappa\tau'-\kappa^3-\tau^3\right) + 4\sqrt{3}v\left(-\kappa\tau'+\tau\kappa'-\kappa^3-\tau^3+\kappa\tau^2+\kappa\tau^2\right) - 3\sqrt{3}v^2\left(\kappa^2-\tau^2\right)(\kappa-\tau)}{6\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}},$$
$$f_{\boldsymbol{Q}} = \frac{\sqrt{2}\left(\kappa^2-\tau^2\right)}{2\sqrt{3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2}}, \qquad g_{\boldsymbol{Q}} = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\boldsymbol{Q}} = \frac{-3\left(\kappa^2-\tau^2\right)^2}{\left(3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right)^2},,$$

$$H_{\boldsymbol{Q}} = \frac{\begin{array}{c} 4\sqrt{3}\left(\tau\kappa'-\kappa\tau'-\kappa^3-\tau^3\right) + \left(\kappa^2-\tau^2\right)(\kappa-\tau) \\ +4\sqrt{3}v\left(-\kappa\tau'+\tau\kappa'-\kappa^3-\tau^3+\kappa\tau^2+\kappa\tau^2\right) - 3\sqrt{3}v^2\left(\kappa^2-\tau^2\right)(\kappa-\tau) \end{array}}{2\left(3\kappa^2+3\tau^2-2\kappa\tau+v2\sqrt{6}(\kappa-\tau)^2+3v^2(\kappa-\tau)^2\right)^{\frac{3}{2}}}.$$

Corollary 3.10. *If α is a circular helix with equal curvatures then the ruled surface $\boldsymbol{Q}(s,v)$ is developable.*

Definition 3.11. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_3}$ moves along on the TNB$-$ Smarandache curve of α. The parametric form of this is given as*

$$\mathcal{M}(s,v) = \frac{1}{\sqrt{3}}(T+N+B) + \frac{v}{\sqrt{2}}(N+B).$$

The first and second partial derivatives of the surface $\mathcal{M}(s,v)$ are given in respective order as follows:

$$\begin{aligned}
\mathcal{M}_s &= \frac{1}{\sqrt{6}}\left(-\left(\sqrt{2}\kappa + v\sqrt{3}\kappa\right)T + \left(\sqrt{2}(\kappa-\tau) - v\sqrt{3}\tau\right)N + \left(\sqrt{2}\tau + v\sqrt{3}\tau\right)B\right)\\
\mathcal{M}_v &= \frac{1}{\sqrt{2}}(N+B), \qquad \mathcal{M}_{vv} = 0, \qquad \mathcal{M}_{sv} = \frac{1}{\sqrt{2}}(-\kappa T - \tau N + \tau B),\\
\mathcal{M}_{ss} &= \frac{1}{\sqrt{6}}\begin{pmatrix} -\left(\sqrt{2}\left(\kappa' + \kappa^2 - \kappa\tau\right) + \sqrt{3}v(\kappa' - \kappa\tau)\right)T \\ +\left(\sqrt{2}\left(\kappa' - \tau' - \kappa^2 - \tau^2\right) + \sqrt{3}v\left(\tau' - \kappa^2 - \tau^2\right)\right)N \\ +\left(\sqrt{2}\left(\tau' + \kappa\tau - \tau^2\right) + \sqrt{3}v\left(\tau' - \tau^2\right)\right)B \end{pmatrix}
\end{aligned}$$

To formulate the normal vector field of $\mathcal{M}(s,v)$ denoted by $N_{\mathcal{M}}$, we first compute

$$\mathcal{M}_s \wedge \mathcal{M}_v = \frac{1}{2\sqrt{6}}\left\{\left(2\kappa - 2\tau\left(2+\sqrt{6}v\right)\right)T + \kappa\left(2+\sqrt{6}v\right)N - \kappa\left(2+\sqrt{6}v\right)B\right\}.$$

When the norm is taken, we have

$$\|\mathcal{M}_s \wedge \mathcal{M}_v\| = \frac{1}{2\sqrt{6}}\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)}.$$

Hence, we obtain

$$N_{\mathcal{M}} = \frac{\left(2\kappa - 4\tau - 2\sqrt{6}\tau v\right)T + \kappa\left(2+\sqrt{6}v\right)N - \kappa\left(2+\sqrt{6}v\right)B}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$\begin{aligned}
E_{\mathcal{M}} &= \frac{1}{6}\left(4\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)\right),\\
F_{\mathcal{M}} &= \frac{\kappa}{\sqrt{6}}, \qquad G_{\mathcal{M}} = 1,\\
e_{\mathcal{M}} &= \frac{-4\sqrt{2}\left(\kappa'\tau - \kappa\tau' - \kappa\tau^2 + \tau\kappa^2 + \kappa^3\right) + 2\sqrt{3}v\left(4\kappa'\tau - 4\kappa\tau^2 + 3\tau\kappa^2 - \kappa\kappa'\right) + v^2 6\sqrt{2}\left(\kappa'\tau - \kappa\tau^2\right)}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)}},\\
f_{\mathcal{M}} &= \frac{-\sqrt{2}\kappa^2}{\sqrt{3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)}}, \qquad g_{\mathcal{M}} = 0.
\end{aligned}$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_{\mathcal{M}} = -\frac{12\kappa^4}{\left(3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)\right)^2},$$

$$H_{\mathcal{M}} = \frac{\begin{array}{c} -12\sqrt{2}\left(\kappa'\tau - \kappa\tau' - \kappa\tau^2 + \tau\kappa^2 + \kappa^3\right) + 2\sqrt{3}\kappa^3 \\ +6\sqrt{3}v\left(4\kappa'\tau - 4\kappa\tau^2 + 3\tau\kappa^2 - \kappa\kappa'\right) + v^2 18\sqrt{2}\left(\kappa'\tau - \kappa\tau^2\right) \end{array}}{\left(3\kappa^2 + 4\tau^2 - 4\kappa\tau + 2\sqrt{6}v\left(\kappa^2 + 2\tau^2 - \kappa\tau\right) + 3v^2\left(\kappa^2 + 2\tau^2\right)\right)^{\frac{3}{2}}}$$

Corollary 3.12. *The ruled surface $\mathcal{M}(s,v)$ cannot be a developable surface.*

Definition 3.13. *Let $\alpha : I \subset R \to R$ be a regular unit speed curve and denote $\{T,\ N,\ B\}$ as its Frenet frame. We define and consider the ruled surface where the unit vector $\vec{r_4}$ moves along on the TNB− Smarandache curve of α. The parametric form of this is given as*

$$\Gamma(s,v) = \frac{1}{\sqrt{3}}(T + N + B) + \frac{v}{\sqrt{3}}(T + N + B).$$

The first and second partial derivatives of the surface $\Gamma(s,v)$ are given in respective order as follows:

$$\begin{aligned}
\Gamma_s &= \frac{1}{\sqrt{3}}(1+v)\left(-\kappa T + (\kappa - \tau)N + \tau B\right),\\
\Gamma_v &= \frac{1}{\sqrt{3}}(T + N + B),\\
\Gamma_{ss} &= \frac{1}{\sqrt{3}}(1+v)\left\{\left(-\kappa' - \kappa^2 + \kappa\tau\right)T + \left(\kappa' - \tau' - \kappa^2 - \tau^2\right)N + \left(\tau' + \kappa\tau - \tau^2\right)B\right\},\\
\Gamma_{sv} &= \frac{1}{\sqrt{3}}\left(-\kappa T + (\kappa - \tau)N + \tau B\right), \qquad \Gamma_{vv} = 0.
\end{aligned}$$

To formulate the normal vector field of $\Gamma(s,v)$ denoted by N_Γ, we first compute

$$\Gamma_s \wedge \Gamma_v = \frac{1}{3}(1+v)\left((\kappa - 2\tau)T + (\kappa + \tau)N + (\tau - 2\kappa)B\right).$$

When the norm is taken, we have

$$\|\Gamma_s \wedge \Gamma_v\| = \frac{\sqrt{6}}{3}(1+v)\sqrt{\kappa^2 - \kappa\tau + \tau^2}.$$

Hence, we obtain

$$N_\Gamma = \frac{(\kappa - 2\tau)T + (\kappa + \tau)N + (\tau - 2\kappa)B}{\sqrt{6}\sqrt{\kappa^2 - \kappa\tau + \tau^2}}.$$

Moreover, from the relations (5) and (6) the coefficients of first and second fundamental forms are calculated as

$$E_\Gamma = \frac{2}{3}(1+v)^2\left(\kappa^2 - \kappa\tau + \tau^2\right), \qquad F_\Gamma = 0, \qquad G_\Gamma = 1,$$

$$e_\Gamma = \frac{(1+v)\left\{-2\left(\kappa^3 + \tau^3\right) - 2\kappa^2\tau + 3\left(\kappa'\tau - \kappa\tau'\right)\right\}}{3\sqrt{2}\sqrt{\kappa^2 - \kappa\tau + \tau^2}}, \qquad f_\Gamma = 0, \qquad g_\Gamma = 0.$$

Finally, by referring the relation (4) the Gaussian and mean curvatures are obtained as

$$K_\Gamma = 0, \qquad H_\Gamma = \frac{-2\left(\kappa^3 + \tau^3\right) - 2\kappa^2\tau + 3\left(\kappa'\tau - \kappa\tau'\right)}{2\sqrt{2}(1+v)\left(\kappa^2 - \kappa\tau + \tau^2\right)^{\frac{3}{2}}}.$$

Remark 3.14. *Note that since the directrix of this ruled surface can be collapsed to a point, it clearly corresponds to a cone and can be parameterized as in the following form:*

$$\Gamma(s,v) = \frac{1}{\sqrt{3}}(1+v)(T+N+B).$$

As known from the literature that for any conical surface, the coefficients F_Γ *and* f_Γ *of first and second fundamental forms in respective order vanish, which corresponds to the relation* $K_\Gamma = 0$*. Therefore, the predefined ruled surface forms always a developable cone. However, we find it worth to do the calculations for validation purposes and providing the relation for mean curvature.*

Example 3.15. *Let us consider the well known Viviani's curve parameterized as*

$$\gamma(t) = \left(a(1+cost), asint, 2asin\frac{1}{2}t\right), \quad t \in [-2\pi, 2\pi], \quad [\textit{Gray, 1997 p.201}].$$

For $a = 0.5$ *and by changing the parameter as* $t = 2s$*, we easily represent the given Viviani's curve as in the following way*

$$\alpha(s) = \left(\cos^2(s), \sin(s)\cos(s), \sin(s)\right), \quad s \in [-\pi, \pi].$$

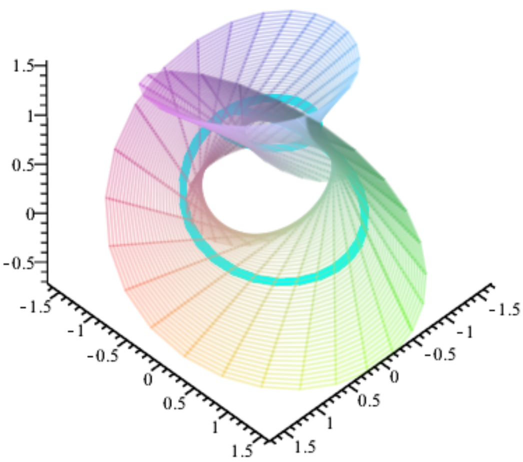
Then, the Frenet apparatus of $\alpha = \alpha(s)$ *are given as*

$$T(s) = \frac{2}{\sqrt{2\cos(2s)+6}}\left(-\sin(2s), \cos(2s), \cos(s)\right),$$

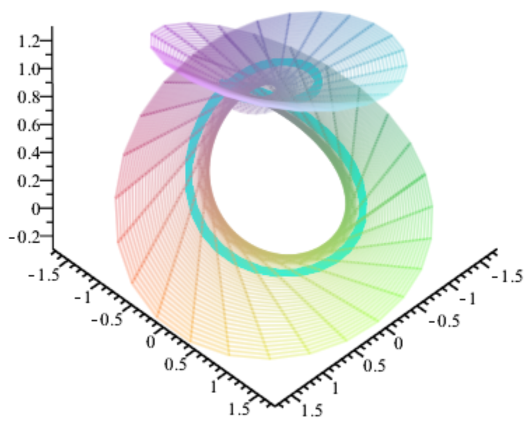
$$N(s) = \frac{-1}{\sqrt{2\cos(2s)+6}\sqrt{6\cos(2s)+26}}\begin{pmatrix} \cos(4s)+12\cos(2s)+3, \\ \sin(4s)+12\sin(2s), \\ 4\sin(s) \end{pmatrix},$$

$$B(s) = \frac{1}{\sqrt{6\cos(2s)+26}}\left(\sin(3s)+3\sin(s), -\cos(3s)-3\cos(s), 4\right).$$

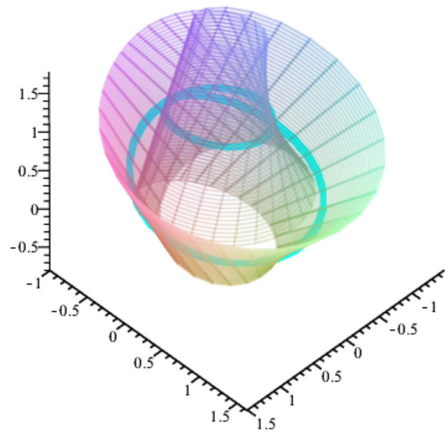
For $s \in [-\pi, \pi]$ and $v \in [-1, 1]$, the ruled surfaces $\mathcal{F}(s,v)$, $\mathcal{U}(s,v)$, $\mathcal{Z}(s,v)$, $\mathcal{S}(s,v)$, $\mathcal{Q}(s,v)$, $\mathcal{M}(s,v)$ and $\Gamma(s,v)$ are sketched in the following figures from (a) to (g).



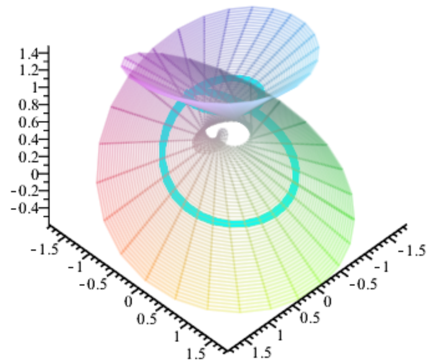
(a) generated by the unit vector $\vec{T}$



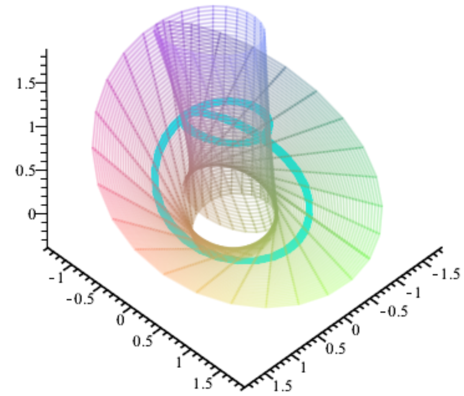
(b) generated by the unit vector $\vec{N}$



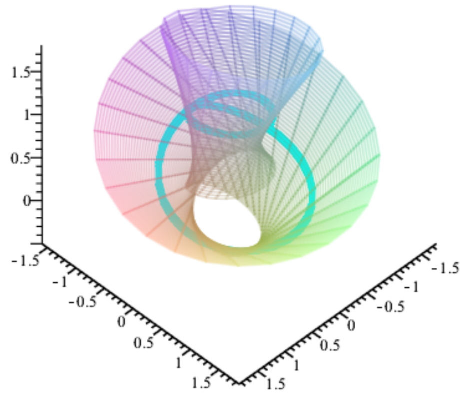
(c) generated by the unit vector $\vec{B}$



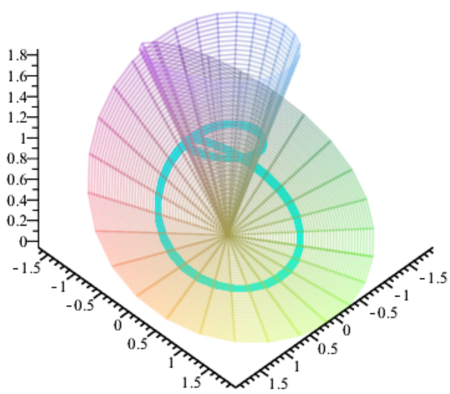
(d) generated by the unit vector $\vec{r_1}$



(e) generated by the unit vector $\vec{r_2}$



(f) generated by the unit vector $\vec{r_3}$



(g) generated by the unit vector $\vec{r_4}$

Figure 1: The ruled surfaces $\mathcal{F}(s,v)$, $\mathcal{U}(s,v)$, $\mathcal{Z}(s,v)$, $\mathcal{S}(s,v)$, $\mathcal{Q}(s,v)$, $\mathcal{M}(s,v)$ and $\Gamma(s,v)$ with the base TNB-Smarandache curve

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