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On the modified orthogonal frames of the non-unit speed curves in Euclidean 3-Space $\mathbb{E}^3$

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Abstract. In this study, the modified frames with both the non-zero curvature and the torsion of the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$ are examined. The relationships between the derivative vectors of the modified frames and the Frenet vectors or the vectors of the modified frames of the curve are given. Besides, the Darboux vectors obtained from the modified orthogonal frames with both the curvature and torsion of the curve and the unit vectors in the direction of these Darboux vectors are investigated. Finally, all these results are shown on the example curves.

1. INTRODUCTION

The Frenet frame, which is a moving frame at a given point on any regular curve in Euclidean 3-space $\mathbb{E}^3$, is one of the most important tools used to analyze the curve. The Frenet frame is an orthonormal frame consisting of the tangent vector, the principal normal vector and binormal vector of the curve. The curvature and the torsion functions can be defined on the curve using this frame. Studies on the Frenet frame of a regular curve in $\mathbb{E}^3$ are available in various sources, such as [5, 9–11, 16, 19, 20]. According to the fundamental theorem of regular curves, a regular curve is a curve with functions $\kappa > 0$ (curvature) and τ (torsion) that can be differentiated at every point of the curve, [6]. However, it is possible for the curvature function to be zero at certain points on the analytical curves. The principal normal and binormal vectors of these curves are generally discontinuous at the zero point of the curvature, that is, the curvature function is not always differentiable. In this case, the Frenet derivative equations of an analytical curve causes ambiguity at a point where the curvature vanishes. Hord and Sasai pondered this problem and discussed another frame that works fine on these points, [12, 17]. In a simple but useful approach, an orthogonal frame was introduced for unit speed analytical curves by Sasai, [18]. Although the vectors of this modified orthogonal frame are obtained by multiplying each Frenet vector by the curvature function κ, they allow the use of a new formula corresponding to the Frenet derivative equations for the above-mentioned case. It is also a useful tool for investigating analytical curves with singular points. Then, Bükcü and Karacan have developed the Sasai's study and they have obtained the newly modified frame through the coefficient of torsion τ by the Frenet vectors and them spherical curves, [3, 4]. There are many studies on the modified orthogonal frame of a curve in Euclidean or Lorentzian 3-space, [1, 2, 7, 8, 14, 15, 21]. Also, the Darboux

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vector of a space curve is the areal velocity vector of the moving frame of the curve. The direction of the Darboux vector is the direction of the instantaneous rotation axis. The Darboux vector can be expressed in terms of the apparatus of the moving frame. The Darboux vector can also be studied in a modified way for space curves with singular points, [13, 22].

In this study, the modified frames with both the non-zero curvature and the torsion of the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$ are examined. The derivatives of the vectors belong to these modified frames are calculated. The relationships between these derivative vectors and the Frenet vectors of the curve or the vectors of the modified frame are given. Besides, the Darboux vectors obtained from the modified orthogonal frames with both the curvature and torsion of the curve and the unit vectors in the direction of these Darboux vectors are investigated. Finally, these all results are investigated on the example curves. The aim of this study is to generalize the modified frame formulas, created for unit speed curves by Sasai, for non-unit speed curves. And thus, it provides ease of operation for the solution of the problem at singular points on a non-unit speed analytic curve, that is, there is no need to transform the curve into a unit speed curve every time.

2. PRELIMINARIES

Let the curve $\alpha(t)$ be a differentiable space curve in $\mathbb{E}^3$. The Frenet vectors, the curvature and the torsion of the curve $\alpha(t)$ are given as follows:

$$T(t) = \frac{\alpha'(t)}{v(t)}, \quad B(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{\|\alpha'(t) \wedge \alpha''(t)\|}, \quad N(t) = B(t) \wedge T(t), \tag{1}$$

$$\kappa(t) = \frac{\|\alpha'(t) \wedge \alpha''(t)\|}{v^3(t)} \qquad and \qquad \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge \alpha''(t)\|^2}, \tag{2}$$

where $v(t) = \|\alpha'(t)\|$, respectively. The Frenet derivative formulas of this curve are as follows:

$$\begin{bmatrix} T'(t) \\ N'(t) \\ B'(t) \end{bmatrix} = \begin{bmatrix} 0 & v(t)\kappa(t) & 0 \\ -v(t)\kappa(t) & 0 & v(t)\tau(t) \\ 0 & -v(t)\tau(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}, \tag{3}$$

[11], [16]. The Darboux vector $W(t)$ of the non-unit speed curve $\alpha(t)$ is as follows:

$$W(t) = N(t) \wedge N'(t) = v(t)(\tau(t)T(t) + \kappa(t)B(t)), \tag{4}$$

where,

$$T'(t) = W(t) \wedge T(t), \quad N'(t) = W(t) \wedge N(t), \quad B'(t) = W(t) \wedge B(t). \tag{5}$$

The unit vector $C(t)$ in direction of the Darboux vector of the non-unit speed curve $\alpha(t)$ is

$$C(t) = \frac{W(t)}{\|W(t)\|} = \frac{v(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}(\tau(t)T(t) + \kappa(t)B(t)), \tag{6}$$

or if the angle between of the binormal vector $B(t)$ and the Darboux vector $W(t)$ of the curve $\alpha(t)$ is $\varphi(t)$, then the unit vector is

$$C(t) = \sin\varphi T(t) + \cos\varphi B(t), \tag{7}$$

[16]. If $v(t) = 1$, then the curve $\alpha(t)$ is called unit speed curve. Let's define a orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ for the unit speed curve $\alpha(t)$ as follows:

$$E_1(t) = \alpha'(t), \quad E_2(t) = E_1'(t), \quad E_3(t) = E_1(t) \wedge E_2(t). \tag{8}$$

If the curvature $\kappa(t)$ of the curve $\alpha(t)$ is non-zero, then there are the following relationships between the vectors $E_1(t), E_2(t), E_3(t)$ and the Frenet vectors of the curve:

$$E_1(t) = T(t), \quad E_2(t) = \kappa(t) N(t), \quad E_3(t) = \kappa(t) B(t), \tag{9}$$

where,

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0, \tag{10}$$

and

$$\langle E_1(t), E_1(t)\rangle = 1, \quad \langle E_2(t), E_2(t)\rangle = \langle E_3(t), E_3(t)\rangle = \kappa^2(t). \tag{11}$$

The orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ is called the modified frame with the curvature $\kappa(t)$ of the unit speed curve $\alpha(t)$. It is noted that the modified orthogonal frame coincides with Frenet frame for $\kappa = 1$. There are the following relationships between the vectors $E_1(t), E_2(t), E_3(t)$ and them derivative vectors:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2(t) & \dfrac{\kappa'(t)}{\kappa(t)} & \tau(t) \\ 0 & -\tau(t) & \dfrac{\kappa'(t)}{\kappa(t)} \end{bmatrix} \begin{bmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{bmatrix}, \tag{12}$$

here, $\tau(t) = \frac{\det(\alpha'(t),\alpha''(t),\alpha'''(t))}{\kappa^2(t)}$ is the torsion of the curve (α), we know that any zero point of $\kappa^2(t)$ is a removable singularity of $\tau(t)$, [17]. Or, let the torsion $\tau(t)$ of the curve $\alpha(t)$ be non-zero. Then let's define the following orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ for the unit speed curve $\alpha(t)$ as follows:

$$A_1(t) = T(t), \quad A_2(t) = \tau(t) N(t), \quad A_3(t) = \tau(t) B(t), \tag{13}$$

where,

$$\langle A_1(t), A_2(t)\rangle = \langle A_2(t), A_3(t)\rangle = \langle A_1(t), A_3(t)\rangle = 0, \tag{14}$$

and

$$\langle A_1(t), A_1(t)\rangle = 1, \quad \langle A_2(t), A_2(t)\rangle = \langle A_3(t), A_3(t)\rangle = \tau^2(t). \tag{15}$$

The orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ is called the modified frame with the torsion $\tau(t)$ of the unit speed curve $\alpha(t)$. And, there are the following relationships between the vectors $A_1(t), A_2(t), A_3(t)$ and them derivative vectors:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\kappa(t)}{\tau(t)} & 0 \\ -\kappa(t)\tau(t) & \dfrac{\tau'(t)}{\tau(t)} & \tau(t) \\ 0 & -\tau(t) & \dfrac{\tau'(t)}{\tau(t)} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix}. \tag{16}$$

The Darboux vector $D(t)$ obtained from the modified orthogonal frame with the curvature $\kappa(t)$ of a unit speed curve $\alpha(t)$ is obtained as follows:

$$D(t) = \tau(t) E_1(t) + E_3(t) \tag{17}$$

and

$$E_2(t) \wedge E_2'(t) = \kappa^2(t) D(t). \tag{18}$$

If the angle between of the Darboux vector $D(t)$ and the vector $E_3(t)$ is $\varphi(t)$, the unit vector in direction of the Darboux vector is

$$G(t) = \sin\varphi E_1(t) + \frac{\cos\varphi}{\kappa(t)} E_3(t). \tag{19}$$

3. The Modified Orthogonal Frames of the Non-Unit Speed Curve in $\mathbb{E}^3$

Let the Frenet frame, the curvature and the torsion of a non-unit speed curve $\alpha(t)$ be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$, respectively.

3.1. The Modified Orthogonal Frame With the Curvature $\kappa(t)$ of a Non-Unit Speed Curve in $\mathbb{E}^3$

Theorem 3.1. *Let the Frenet frame and the curvature be $\{T(t), N(t), B(t)\}$ and $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. The modified orthogonal frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of the curve $\alpha(t)$ is as follows:*

$$\begin{cases} E_1(t) = v(t)\, T(t), \\ E_2(t) = v^2(t)\, \kappa(t)\, N(t), \\ E_3(t) = v^3(t)\, \kappa(t)\, B(t). \end{cases} \tag{20}$$

Proof. Let's create the vectors $E_1(t), E_2(t), E_3(t)$ using the Gram-Schmidt orthogonalization procedure as follows:

$$\begin{cases} E_1(t) = \alpha'(t), \\ E_2(t) = E_1'(t) - \dfrac{\left\langle E_1'(t), E_1(t)\right\rangle}{\langle E_1(t), E_1(t)\rangle} E_1(t), \\ E_3(t) = E_1(t) \wedge E_2(t). \end{cases} \tag{21}$$

Here, since the curve $\alpha(t)$ is not an unit speed curve, we can't use the expression (8). From the expression (1), the vector $E_1(t)$ is obtained as follows:

$$E_1(t) = v(t)\, T(t). \tag{22}$$

From the expressions (3) and (22), the following equation is gotten:

$$\frac{\left\langle E_1'(t), E_1(t)\right\rangle}{\langle E_1(t), E_1(t)\rangle} = \frac{v'(t)}{v(t)}. \tag{23}$$

From the expressions (3), (21) and (23), the vectors $E_2(t)$ and $E_3(t)$ are obtained as follows:

$$E_2(t) = v^2(t)\, \kappa(t)\, N(t), \tag{24}$$

$$E_3(t) = v^3(t)\, \kappa(t)\, B(t). \tag{25}$$

The proof is completed from the expressions (22), (24) and (25). □

Corollary 3.2. *As a result of Theorem 3.1, the following equations are obtained for the vectors $E_1(t), E_2(t), E_3(t)$:*

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0, \tag{26}$$

$$\begin{cases} \langle E_1(t), E_1(t)\rangle = v^2(t), \\ \langle E_2(t), E_2(t)\rangle = v^4(t)\, \kappa^2(t), \\ \langle E_3(t), E_3(t)\rangle = v^6(t)\, \kappa^2(t), \end{cases} \tag{27}$$

and

$$\begin{cases} E_1(t) \wedge E_2(t) = E_3(t), \\ E_2(t) \wedge E_3(t) = v^4(t)\kappa^2(t) E_1(t), \\ E_3(t) \wedge E_1(t) = v^2(t) E_2(t). \end{cases} \tag{28}$$

Remark 3.3. *The modified frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of the non-unit speed curve $\alpha(t)$ is indeed orthogonal (from the expression (26)), but is not orthonormal (from the expression (ref20)), because the vectors $E_1(t), E_2(t), E_3(t)$ are not unit vectors (if not $v(t) = \kappa(t) = 1$ at the same time). If $v(t) = \kappa(t) = 1$ at the same time, then the modified frame $\{E_1(t), E_2(t), E_3(t)\}$ becomes an orthonormal frame.*

Theorem 3.4. *Let the Frenet frame, the curvature and the torsion be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. And let the modified orthogonal frame with the curvature $\kappa(t)$ of the curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. There are the following equations between the Frenet vectors and the derivative vectors $E_1'(t), E_2'(t), E_3'(t)$:*

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} v'(t) & v^2(t)\kappa(t) & 0 \\ -v^3(t)\kappa^2(t) & 2v(t)v'(t)\kappa(t) + v^2(t)\kappa'(t) & v^3(t)\kappa(t)\tau(t) \\ 0 & -v^4(t)\kappa(t)\tau(t) & 3v^2(t)v'(t)\kappa(t) + v^3(t)\kappa'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \quad (29)$$

Proof. By using the expression (3), from the expression (20), we obtain the derivative vectors $E_1'(t)$, $E_2'(t)$, $E_3'(t)$ in terms of the Frenet vectors $T(t), N(t), B(t)$ as follows:

$$\begin{aligned} E_1'(t) &= (v(t)T(t))', \\ E_1'(t) &= v'(t)T(t) + v^2(t)\kappa(t)N(t), \end{aligned} \quad (30)$$

$$\begin{aligned} E_2'(t) &= \left(v^2(t)\kappa(t)N(t)\right)', \\ E_2'(t) &= -v^3(t)\kappa^2(t)T(t) + \left(2v(t)v'(t)\kappa(t) + v^2(t)\kappa'(t)\right)N(t) + v^3(t)\kappa(t)\tau(t)B(t), \end{aligned} \quad (31)$$

$$\begin{aligned} E_3'(t) &= \left(v^3(t)\kappa(t)B(t)\right)' \\ E_3'(t) &= -v^4(t)\kappa(t)\tau(t)N(t) + \left(3v^2(t)v'(t)\kappa(t) + v^3(t)\kappa'(t)\right)B(t). \end{aligned} \quad (32)$$

The proof is completed from the expressions (30), (31) and (32). □

Theorem 3.5. *Let the Frenet frame, the curvature and the torsion be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. And let the modified orthogonal frame with the curvature $\kappa(t)$ of the curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. There are the following equations between their derivatives and the vectors $E_1(t), E_2(t), E_3(t)$:*

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{v'(t)}{v(t)} & 1 & 0 \\ -v^2(t)\kappa^2(t) & \dfrac{2v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} & \tau(t) \\ 0 & -v^2(t)\tau(t) & \dfrac{3v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} \end{bmatrix} \begin{bmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{bmatrix}. \quad (33)$$

Proof. From the expression (20), the Frenet vectors $T(t), N(t), B(t)$ are written in terms of the vectors $E_1(t), E_2(t), E_3(t)$ as follows:

$$T(t) = \frac{E_1(t)}{v(t)}, \quad N(t) = \frac{E_2(t)}{v^2(t)\kappa(t)}, \quad B(t) = \frac{E_3(t)}{v^3(t)\kappa(t)}. \quad (34)$$

If the expression (34) is substituted in the expressions (30), (31) and (32), respectively, we get

$$E_1'(t) = \frac{v'(t)}{v(t)}E_1(t) + E_2(t), \tag{35}$$

$$E_2'(t) = -v^2(t)\kappa^2(t)E_1(t) + \left(\frac{2v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)}\right)E_2(t) + \tau(t)E_3(t), \tag{36}$$

$$E_3'(t) = -v^2(t)\tau(t)E_2(t) + \left(\frac{3v'(t)}{v(t)} + \frac{\kappa'(t)}{\kappa(t)}\right)E_3(t). \tag{37}$$

The proof is completed from the expressions (35), (36) and (37). □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, from the expressions (20) and (27), the expressions (9) and (11) are obtained, respectively. And the expression (28) would be as follows:

$$E_1(t) \wedge E_2(t) = E_3(t), \quad E_2(t) \wedge E_3(t) = \kappa^2(t)E_1(t), \quad E_3(t) \wedge E_1(t) = E_2(t). \tag{38}$$

Also, from the expression (33), the following equations are obtained:

$$\begin{bmatrix} E_1'(t) \\ E_2'(t) \\ E_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa^2(t) & \kappa'(t) & \kappa(t)\tau(t) \\ 0 & -\kappa(t)\tau(t) & \kappa'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{39}$$

Finally, we obtain the equalities between of the modified orthogonal frame and the derivative vectors of the modified orthogonal frame with the curvature $\kappa(t)$ of the unit speed curve $\alpha(t)$, like the expression (12).

3.2. *The Modified Orthogonal Frame With the Torsion $\tau(t)$ of a Non-Unit Speed Curve in $\mathbb{E}^3$*

Theorem 3.6. *Let the Frenet frame and the curvature be $\{T(t), N(t), B(t)\}$ and $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. The modified orthogonal frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of the curve $\alpha(t)$ is as follows:*

$$\begin{cases} A_1(t) = v(t)T(t), \\ A_2(t) = v^2(t)\tau(t)N(t), \\ A_3(t) = v^3(t)\tau(t)B(t). \end{cases} \tag{40}$$

Proof. Let's create the vectors $A_1(t), A_2(t), A_3(t)$ using the Gram-Schmidt orthogonalization procedure as follows:

$$\begin{cases} A_1(t) = \alpha'(t), \\ A_2(t) = \dfrac{\tau(t)}{\kappa(t)}\left(A_1'(t) - \dfrac{\left\langle A_1'(t),\, A_1(t)\right\rangle}{\langle A_1(t),\, A_1(t)\rangle}A_1(t)\right), \\ A_3(t) = A_1(t) \wedge A_2(t). \end{cases} \tag{41}$$

Here, since the curve $\alpha(t)$ is not an unit speed curve, we can't use the expression (13). From the expression (1), the vector $A_1(t)$ is obtained as follows:

$$A_1(t) = v(t)T(t). \tag{42}$$

From the expression (22), we see that the vectors $E_1(t)$ and $A_1(t)$ are equal. So from the expression (23), we get

$$\frac{\left\langle A_1'(t), A_1(t)\right\rangle}{\left\langle A_1(t), A_1(t)\right\rangle} = \frac{v'(t)}{v(t)}. \tag{43}$$

From the expressions (3), (42) and (43), the vectors $A_2(t)$ and $A_3(t)$ are obtained as follows:

$$A_2(t) = v^2(t)\,\tau(t)\,N(t), \tag{44}$$

$$A_3(t) = v^3(t)\,\tau(t)\,B(t). \tag{45}$$

The proof is completed from the expressions (42), (44) and (45). □

Corollary 3.7. *As a result of Theorem 3.6, the following equations are obtained for the vectors $A_1(t), A_2(t), A_3(t)$:*

$$\langle A_1(t), A_2(t)\rangle = \langle A_2(t), A_3(t)\rangle = \langle A_1(t), A_3(t)\rangle = 0, \tag{46}$$

$$\begin{cases} \langle A_1(t), A_1(t)\rangle = v^2(t), \\ \langle A_2(t), A_2(t)\rangle = v^4(t)\,\tau^2(t), \\ \langle A_3(t), A_3(t)\rangle = v^6(t)\,\tau^2(t), \end{cases} \tag{47}$$

and

$$\begin{cases} A_1(t) \wedge A_2(t) = A_3(t), \\ A_2(t) \wedge A_3(t) = v^4(t)\tau^2(t)A_1(t), \\ A_3(t) \wedge A_1(t) = v^2(t)A_2(t). \end{cases} \tag{48}$$

Remark 3.8. *The modified frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of the non-unit speed curve $\alpha(t)$ is indeed orthogonal (from the expression (46)), but is not orthonormal (from the expression (47)), because the vectors $A_1(t), A_2(t), A_3(t)$ are not unit vectors (if not $v(t) = \tau^2(t) = 1$ at the same time). If $v(t) = \tau^2(t) = 1$ at the same time, then the modified frame $\{A_1(t), A_2(t), A_3(t)\}$ becomes an orthonormal frame.*

Theorem 3.9. *Let the Frenet frame, the curvature and the torsion be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. And let the modified orthogonal frame with the torsion $\tau(t)$ of the curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. There are the following equations between the Frenet vectors and the derivative vectors $A_1'(t), A_2'(t), A_3'(t)$:*

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} v'(t) & v^2(t)\,\kappa(t) & 0 \\ -v^3(t)\,\kappa(t)\,\tau(t) & 2v(t)\,v'(t)\,\tau(t) + v^2(t)\,\tau'(t) & v^3(t)\,\tau^2(t) \\ 0 & -v^4(t)\,\tau^2(t) & 3v^2(t)\,v'(t)\,\tau(t) + v^3(t)\,\tau'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}. \tag{49}$$

Proof. By using the expression (3), from the expression (40), we obtain the derivative vectors $A_1'(t)$, $A_2'(t)$,

$A_3'(t)$ in terms of the Frenet vectors $T(t)$, $N(t)$, $B(t)$ as follows:

$$\begin{aligned} A_1'(t) &= (v(t)T(t))', \\ A_1'(t) &= v'(t)T(t) + v^2(t)\kappa(t)N(t), \end{aligned} \tag{50}$$

$$\begin{aligned} A_2'(t) &= \left(v^2(t)\tau(t)N(t)\right)' \\ A_2'(t) &= \left(-v^3(t)\kappa(t)\tau(t)\right)T(t) + \left(2v(t)v'(t)\tau(t) + v^2(t)\tau'(t)\right)N(t) + v^3(t)\tau^2(t)B(t), \end{aligned} \tag{51}$$

$$\begin{aligned} A_3'(t) &= \left(v^3(t)\tau(t)B(t)\right)' \\ A_3'(t) &= -v^4(t)\tau^2(t)N(t) + \left(3v^2(t)v'(t)\tau(t) + v^3(t)\tau'(t)\right)B(t). \end{aligned} \tag{52}$$

The proof is completed from the expressions (50), (51) and (52). □

Theorem 3.10. *Let the Frenet frame, the curvature and the torsion be $\{T(t), N(t), B(t)\}$, $\kappa(t)$ and $\tau(t)$ of the non-unit speed space curve $\alpha(t)$, respectively. And let the modified orthogonal frame with the torsion $\tau(t)$ of the curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. There are the following equations between their derivatives and the vectors $A_1(t), A_2(t), A_3(t)$:*

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} \dfrac{v'(t)}{v(t)} & \dfrac{\kappa(t)}{\tau(t)} & 0 \\ -v^2(t)\kappa(t)\tau(t) & \dfrac{2v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)} & \tau(t) \\ 0 & -v^2(t)\tau(t) & \dfrac{3v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)} \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \end{bmatrix}. \tag{53}$$

Proof. From the expression (40), the Frenet vectors $T(t), N(t), B(t)$ are written in terms of the vectors $A_1(t), A_2(t), A_3(t)$ as follows:

$$T(t) = \frac{A_1(t)}{v(t)}, \quad N(t) = \frac{A_2(t)}{v^2(t)\tau(t)}, \quad B(t) = \frac{A_3(t)}{v^3(t)\tau(t)}. \tag{54}$$

If the expression (54) is substituted in the expressions (50), (51) and (52), respectively, we get

$$A_1'(t) = \frac{v'(t)}{v(t)}A_1(t) + \frac{\kappa(t)}{\tau(t)}A_2(t), \tag{55}$$

$$A_2'(t) = -v^2(t)\kappa(t)\tau(t)A_1(t) + \left(\frac{2v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)}\right)A_2(t) + \tau(t)A_3(t), \tag{56}$$

$$A_3'(t) = -v^2(t)\tau(t)A_2(t) + \left(\frac{3v'(t)}{v(t)} + \frac{\tau'(t)}{\tau(t)}\right)A_3(t). \tag{57}$$

The proof is completed from the expressions (55), (56) and (57). □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expressions (40) and (47), the expressions (13) and (15) are obtained, respectively. And the expression (48) would be as follows:

$$A_1(t) \wedge A_2(t) = A_3(t), \quad A_2(t) \wedge A_3(t) = \tau^2(t)A_1(t), \quad A_3(t) \wedge A_1(t) = A_2(t). \tag{58}$$

Also, from the expression (49), the following equations are obtained:

$$\begin{bmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t)\tau(t) & \tau'(t) & \tau^2(t) \\ 0 & -\tau^2(t) & \tau'(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \\ B(t) \end{bmatrix}.$$

Finally, we obtain the equalities between of the modified orthogonal frame and the derivative vectors of the modified orthogonal frame with the torsion $\tau(t)$ of the unit speed curve $\alpha(t)$, like the expression (16).

4. The Darboux vectors obtained from the modified orthogonal frames the non-unit speed curves in Euclidean 3-space $\mathbb{E}^3$

In this section, we will calculate the equivalent of the Darboux vector $W(t)$ (or the unit vector in the direction of the Darboux vector $C(t)$) obtained from the Frenet frame in terms of the vectors of the modified frame of a non-unit speed curve $\alpha(t)$. But to avoid confusion, we will denote the Darboux vector (the unit vector in the direction of the Darboux vector) obtained from the modified frame with $D(t)$ (or $G(t)$).

4.1. *The Darboux vector obtained from the modified orthogonal frame with the curvature* $\kappa(t)$ *of a non-unit speed curve in Euclidean 3-space* $\mathbb{E}^3$

Theorem 4.1. *Let the modified orthogonal frame with the curvature $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. The Darboux vector $D(t)$ obtained from this frame is as follows:*

$$D(t) = \tau(t) E_1(t) + \frac{E_3(t)}{v^2(t)}, \tag{59}$$

here,

$$\left(\frac{E_1(t)}{v(t)}\right)' = D(t) \wedge \frac{E_1(t)}{v(t)} = \frac{E_2(t)}{v(t)}, \tag{60}$$

$$\left(\frac{E_2(t)}{v^2(t)\kappa(t)}\right)' = D(t) \wedge \frac{E_2(t)}{v^2(t)\kappa(t)} = -\kappa(t) E_1(t) + \frac{\tau(t)}{v^2(t)\kappa(t)} E_3(t), \tag{61}$$

$$\left(\frac{E_3(t)}{v^3(t)\kappa(t)}\right)' = D(t) \wedge \frac{E_3(t)}{v^3(t)\kappa(t)} = -\frac{\tau(t)}{v(t)\kappa(t)} E_2(t). \tag{62}$$

Proof. From the expression (4) and (34), the Darboux vector $D(t)$ is obtained as the expression (59). Also, from the expression (33), we can write the following equations:

$$\left(\frac{E_1(t)}{v(t)}\right)' = \frac{1}{v(t)} E_1'(t) - \frac{v'(t)}{v^2(t)} E_1(t), \tag{63}$$

$$\left(\frac{E_1(t)}{v(t)}\right)' = \frac{E_2(t)}{v(t)}. \tag{64}$$

On the other hand, from the expression (28), we get

$$D(t) \wedge \frac{E_1(t)}{v(t)} = \left(\tau(t) E_1(t) + \frac{E_3(t)}{v^2(t)}\right) \wedge \frac{E_1(t)}{v(t)},$$

$$D(t) \wedge \frac{E_1(t)}{v(t)} = \frac{E_2(t)}{v(t)}. \tag{65}$$

From the equality of the expressions (64) and (65), the expression (60) is gotten. If similar operations are applied for $E_2(t)$ and $E_3(t)$ vectors, the equations (61) and (62) are obtained. □

Corollary 4.2. *As a result of Theorem 4.1, the following equations are obtained:*

$$\begin{cases} E_1'(t) = D(t) \wedge E_1(t) + \dfrac{v'(t)}{v(t)} E_1(t), \\ \\ E_2'(t) = D(t) \wedge E_2(t) + \left(\dfrac{2v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} \right) E_2(t), \\ \\ E_3'(t) = D(t) \wedge E_3(t) + \left(\dfrac{3v'(t)}{v(t)} + \dfrac{\kappa'(t)}{\kappa(t)} \right) E_3(t). \end{cases} \tag{66}$$

Corollary 4.3. *As a result of Corollary 4.2, the following equations are obtained:*

$$\begin{cases} E_1(t) \wedge E_1'(t) = E_3(t), \\ \\ E_2(t) \wedge E_2'(t) = v^4(t)\kappa^2(t) D(t), \\ \\ E_3(t) \wedge E_3'(t) = v^6(t)\kappa^2(t)\tau(t) E_1(t). \end{cases} \tag{67}$$

Corollary 4.4. *From Corollary 4.3, the following equation are gotten:*

$$D(t) = \frac{E_2(t) \wedge E_2'(t)}{v^4(t)\kappa^2(t)}. \tag{68}$$

Remark 4.5. *Here, we have actually expressed the Darboux vector $W(t)$ of the Frenet frame of the non-unit speed curve $\alpha(t)$ in terms of the modified frame $\{E_1(t), E_2(t), E_3(t)\}$ with the curvature $\kappa(t)$ of the curve, with the vector $D(t)$ in the expression (68).*

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expression (59) is obtained as in expression (17). And the expressions (66) and (67) would be as following:

$$\begin{cases} E_1'(t) = D(t) \wedge E_1(t), \\ \\ E_2'(t) = D(t) \wedge E_2(t) + \dfrac{\kappa'(t)}{\kappa(t)} E_2(t), \\ \\ E_3'(t) = D(t) \wedge E_3(t) + \dfrac{\kappa'(t)}{\kappa(t)} E_3(t) \end{cases}$$

and

$$E_1(t) \wedge E_1'(t) = E_3(t), \quad E_2(t) \wedge E_2'(t) = \kappa^2(t) D(t), \quad E_3(t) \wedge E_3'(t) = \kappa^2(t)\tau(t) E_1(t).$$

Finally, the expression (68) is obtained as in the expression (18).

Theorem 4.6. *Let the modified orthogonal frame with the curvature $\kappa(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{E_1(t), E_2(t), E_3(t)\}$. And let the angle between the Darboux vector $D(t)$ and the vector $E_3(t)$ be $\varphi(t)$. The unit vector in the direction of the Darboux vector $G(t)$ is as follows:*

$$G(t) = \frac{\sin\varphi}{v(t)} E_1(t) + \frac{\cos\varphi}{v^3(t)\kappa(t)} E_3(t). \tag{69}$$

Proof. From the expressions (27) and (59), we get

$$\begin{aligned} \|D(t)\| &= \sqrt{\langle E_1(t), E_1(t)\rangle \tau^2(t) + \frac{\langle E_3(t), E_3(t)\rangle}{v^4(t)}}, \\ \|D(t)\| &= v(t)\sqrt{\kappa^2(t) + \tau^2(t)}. \end{aligned} \tag{70}$$

From the expressions (59) and (70), the unit vector in the direction of the Darboux vector $G(t)$ is gotten as follows:

$$G(t)=\frac{\tau(t)}{v(t)\sqrt{\kappa^2(t)+\tau^2(t)}}E_1(t)+\frac{1}{v^3(t)\sqrt{\kappa^2(t)+\tau^2(t)}}E_3(t)\,. \tag{71}$$

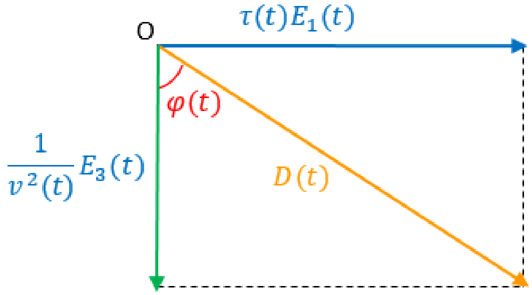

Figure 1: The Darboux vector obtained from the modified orthogonal frame with the curvature $\kappa(t)$

If the angle between of the Darboux vector $D(t)$ and the vector $E_3(t)$ is $\varphi(t)$, from the Figure 1 and the expressions (27) and (70), we write

$$\cos\varphi=\frac{\kappa(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}\quad\text{and}\quad\sin\varphi=\frac{\tau(t)}{\sqrt{\kappa^2(t)+\tau^2(t)}}. \tag{72}$$

From the expressions (71) and (72), the expression (69) is obtained. □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t)=1$. In this case, the expression (69) is obtained as in expression (19).

4.2. *The Darboux vector obtained from the modified orthogonal frame with the torsion* $\tau(t)$ *of a non-unit speed curve in Euclidean 3-space* $\mathbb{E}^3$

Theorem 4.7. *Let the modified orthogonal frame with the torsion $\tau(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{A_1(t),A_2(t),A_3(t)\}$. The Darboux vector $\overline{D}(t)$ obtained from this frame is as follows:*

$$\overline{D}(t)=\tau(t)A_1(t)+\frac{\kappa(t)}{v^2(t)\tau(t)}A_3(t), \tag{73}$$

here,

$$\left(\frac{A_1(t)}{v(t)}\right)'=\overline{D}(t)\wedge\frac{A_1(t)}{v(t)}=\frac{\kappa(t)}{v(t)\tau(t)}A_2(t), \tag{74}$$

$$\left(\frac{A_2(t)}{v^2(t)\tau(t)}\right)'=\overline{D}(t)\wedge\frac{A_2(t)}{v^2(t)\tau(t)}=-\kappa(t)A_1(t)+\frac{A_3(t)}{v^2(t)}, \tag{75}$$

$$\left(\frac{A_3(t)}{v^3(t)\tau(t)}\right)'=\overline{D}(t)\wedge\frac{A_3(t)}{v^3(t)\tau(t)}=-\frac{A_2(t)}{v(t)}. \tag{76}$$

Proof. From the expressions (4) and (54), the Darboux vector $\overline{D}(t)$ is obtained as the expression (73). Also, from the expression (53), we can write the following equations:

$$\left(\frac{A_1(t)}{v(t)}\right)'=\frac{1}{v(t)}A_1'(t)-\frac{v'(t)}{v^2(t)}A_1(t), \tag{77}$$

$$\left(\frac{A_1(t)}{v(t)}\right)'=\frac{\kappa(t)}{v(t)\tau(t)}A_2(t)\,. \tag{78}$$

On the other hand, from the expression (48), we get

$$\begin{aligned}
\overline{D}(t) \wedge \frac{A_1(t)}{v(t)} &= \left(\tau(t) A_1(t) + \frac{\kappa(t)}{v^2(t)\,\tau(t)} A_3(t)\right) \wedge \frac{A_1(t)}{v(t)}, \\
\overline{D}(t) \wedge \frac{A_1(t)}{v(t)} &= \frac{\kappa(t)}{v(t)\,\tau(t)} A_2(t). \qquad (79)
\end{aligned}$$

From the equality of the expressions (78) and (79), the expression (74) is gotten. If similar operations are applied for $A_2(t)$ and $A_3(t)$ vectors, the equations (75) and (76) are obtained. □

Corollary 4.8. *As a result of Theorem 4.7, the following equations are obtained:*

$$\begin{cases}
A_1'(t) = \overline{D}(t) \wedge A_1(t) + \dfrac{v'(t)}{v(t)} A_1(t), \\[2ex]
A_2'(t) = \overline{D}(t) \wedge A_2(t) + \left(\dfrac{2v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)}\right) A_2(t), \\[2ex]
A_3'(t) = \overline{D}(t) \wedge A_3(t) + \left(\dfrac{3v'(t)}{v(t)} + \dfrac{\tau'(t)}{\tau(t)}\right) A_3(t).
\end{cases} \qquad (80)$$

Corollary 4.9. *As a result of Corollary 4.8, the following equations are obtained:*

$$\begin{cases}
A_1(t) \wedge A_1'(t) = \dfrac{\kappa(t)}{\tau(t)} A_3(t), \\[2ex]
A_2(t) \wedge A_2'(t) = v^4(t)\,\tau^2(t)\,\overline{D}(t), \\[2ex]
A_3(t) \wedge A_3'(t) = v^6(t)\,\tau^3(t)\,A_1(t).
\end{cases} \qquad (81)$$

Corollary 4.10. *From Corollary 4.9, the following equation is gotten:*

$$\overline{D}(t) = \frac{A_2(t) \wedge A_2'(t)}{v^4(t)\,\tau^2(t)}. \qquad (82)$$

Remark 4.11. *Here, we have actually expressed the Darboux vector $W(t)$ of the Frenet frame of the non-unit speed curve $\alpha(t)$ in terms of the modified frame $\{A_1(t), A_2(t), A_3(t)\}$ with the torsion $\tau(t)$ of the curve, with the vector $\overline{D}(t)$ in the expression (82).*

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expressions (73), (80), (81) and (82) would be as following:

$$\overline{D}(t) = \tau(t) A_1(t) + \frac{\kappa(t)}{\tau(t)} A_3(t),$$

$$\begin{cases}
A_1'(t) = \overline{D}(t) \wedge A_1(t), \\[2ex]
A_2'(t) = \overline{D}(t) \wedge A_2(t) + \dfrac{\tau'(t)}{\tau(t)} A_2(t), \\[2ex]
A_3'(t) = \overline{D}(t) \wedge A_3(t) + \dfrac{\tau'(t)}{\tau(t)} A_3(t),
\end{cases}$$

$$A_1(t) \wedge A_1'(t) = \frac{\kappa(t)}{\tau(t)} A_3(t), \quad A_2(t) \wedge A_2'(t) = \tau^2(t)\overline{D}(t), \quad A_3(t) \wedge A_3'(t) = \tau^3(t) A_1(t)$$

and

$$\overline{D}(t) = \frac{A_2(t) \wedge A_2'(t)}{\tau^2(t)}.$$

Theorem 4.12. *Let the modified orthogonal frame with the torsion $\tau(t)$ of the non-unit speed space curve $\alpha(t)$ be $\{A_1(t), A_2(t), A_3(t)\}$. And let the angle between the Darboux vector $\overline{D}(t)$ and the vector $A_3(t)$ be $\overline{\varphi}(t)$. The unit vector in the direction of the Darboux vector $\overline{G}(t)$ is as follows:*

$$\overline{G}(t) = \frac{\sin\overline{\varphi}}{v(t)} A_1(t) + \frac{\cos\overline{\varphi}}{v^3(t)\tau(t)} A_3(t). \tag{83}$$

Proof. From the expressions (47) and (73), we get

$$\begin{aligned} \left\|\overline{D}(t)\right\| &= \sqrt{\langle A_1(t), A_1(t)\rangle \tau^2(t) + \frac{\langle A_3(t), A_3(t)\rangle \kappa^2(t)}{v^4(t)\tau^2(t)}}, \\ \left\|\overline{D}(t)\right\| &= v(t)\sqrt{\kappa^2(t) + \tau^2(t)}. \end{aligned} \tag{84}$$

From the expressions (73) and (84), the unit vector in the direction of the Darboux vector $\overline{G}(t)$ is gotten as follows:

$$\overline{G}(t) = \frac{\tau(t)}{v(t)\sqrt{\kappa^2(t) + \tau^2(t)}} A_1(t) + \frac{\kappa(t)}{v^3(t)\tau(t)\sqrt{\kappa^2(t) + \tau^2(t)}} A_3(t). \tag{85}$$

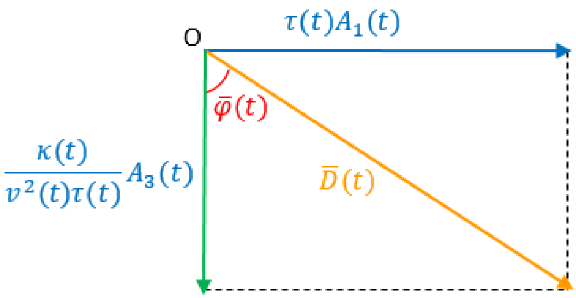

Figure 2: The Darboux vector obtained from the modified orthogonal frame with the torsion $\tau(t)$

If the angle between of the Darboux vector $\overline{D}(t)$ and the vector $A_3(t)$ is $\overline{\varphi}(t)$, from the Figure 2 and the expressions (47) and (84), we write

$$\cos\overline{\varphi} = \frac{\kappa(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}} \quad \text{and} \quad \sin\overline{\varphi} = \frac{\tau(t)}{\sqrt{\kappa^2(t) + \tau^2(t)}}. \tag{86}$$

From the expressions (85) and (86), the expression (83) is obtained. □

Special Case: If the curve $\alpha(t)$ is an unit speed curve, $v(t) = 1$. In this case, the expression (83) is obtained as follows:

$$\overline{G}(t) = \sin\overline{\varphi} A_1(t) + \frac{\cos\overline{\varphi}}{\tau(t)} A_3(t). \tag{87}$$

Example 4.13. *Let's consider the non-unit speed Euler spiral (clothoid or Cornu spiral)*

$$\alpha(t)=\left(a\oint_0^t \cos\left(\frac{\pi x^2}{2}\right)dx,\ a\oint_0^t \sin\left(\frac{\pi x^2}{2}\right)dx,\ at\right),$$

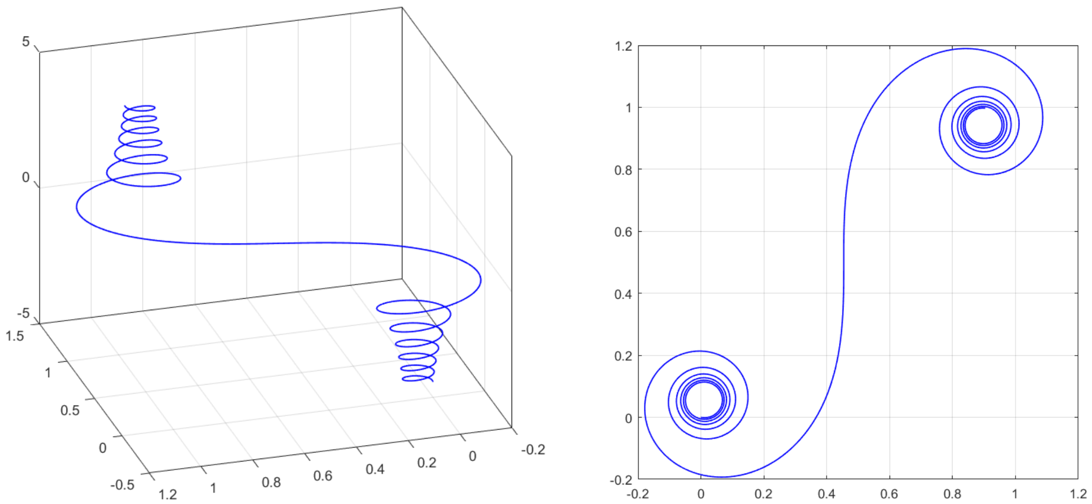

Figure 3: Euler spiral

Figure 3, [9]. Here the components $\oint_0^t \cos\left(\frac{\pi x^2}{2}\right)dx$ *and* $\oint_0^t \sin\left(\frac{\pi x^2}{2}\right)dx$ *are called Fresnel integrals. Then the first, second and third derivative vectors of* $\alpha(t)$ *are as follows:*

$$\alpha'(t)=\left(a\cos\left(\frac{\pi t^2}{2}\right),\ a\sin\left(\frac{\pi t^2}{2}\right),\ a\right),$$

$$\alpha''(t)=\left(-a\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a\pi t\cos\left(\frac{\pi t^2}{2}\right),\ 0\right),$$

$$\alpha'''(t)=\left(-a\pi^2t^2\cos\left(\frac{\pi t^2}{2}\right), -a\pi^2t^2\sin\left(\frac{\pi t^2}{2}\right),\ 0\right).$$

Here, $\|\alpha'(t)\|=v(t)=\sqrt{2}\,|a|$, *if* $a\neq\pm\frac{1}{\sqrt{2}}$, *the curve be* $\alpha(t)$ *is not an unit speed curve. So the Frenet vectors, the curvature and the torsion are obtained as follows:*

$$\begin{aligned}
T(t) &= \left(\frac{a}{\sqrt{2}\,|a|}\cos\left(\frac{\pi t^2}{2}\right), \frac{a}{\sqrt{2}\,|a|}\sin\left(\frac{\pi t^2}{2}\right), \frac{a}{\sqrt{2}\,|a|}\right),\\
N(t) &= \left(-\frac{at}{|a|\,|t|}\sin\left(\frac{\pi t^2}{2}\right), \frac{at}{|a|\,|t|}\cos\left(\frac{\pi t^2}{2}\right), 0\right),\\
B(t) &= \left(-\frac{t}{\sqrt{2}\,|t|}\cos\left(\frac{\pi t^2}{2}\right), -\frac{t}{\sqrt{2}\,|t|}\sin\left(\frac{\pi t^2}{2}\right), \frac{t}{\sqrt{2}\,|t|}\right),
\end{aligned}$$

$$\kappa(t)=\frac{\pi\,|t|}{2\,|a|}\quad,\quad \tau(t)=\frac{\pi t}{2a}.$$

Now, let's examine the left and right limits of the vectors $N(t)$ and $B(t)$:

$$\lim_{t\to 0^+} N(t) = \left(0, \frac{a}{|a|}, 0\right) \quad \text{and} \quad \lim_{t\to 0^-} N(t) = \left(0, -\frac{a}{|a|}, 0\right),$$

$$\lim_{t\to 0^+} B(t) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \lim_{t\to 0^-} B(t) = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right).$$

Since $\lim_{t\to 0^+} N(t) \neq \lim_{t\to 0^-} N(t)$ and $\lim_{t\to 0^+} B(t) \neq \lim_{t\to 0^-} B(t)$, there is no limit at $t = 0$. So the normal vector and binormal vector are discontinuous at $t = 0$. And it is clear that, the curvature function is not differentiable at $t = 0$. Then, to prevent the occurrence of two reverse oriented principal normal vector and binormal vector, it is helpful to use the following modified orthogonal frame with the curvature $\kappa(t)$, obtained from Frenet vectors:

$$\begin{aligned} E_1(t) &= \left(a\cos\left(\frac{\pi t^2}{2}\right),\ a\sin\left(\frac{\pi t^2}{2}\right),\ a\right), \\ E_2(t) &= \left(-a\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a\pi t\cos\left(\frac{\pi t^2}{2}\right),\ 0\right), \\ E_3(t) &= \left(-a^2\pi t\cos\left(\frac{\pi t^2}{2}\right),\ -a^2\pi t\sin\left(\frac{\pi t^2}{2}\right),\ a^2\pi t\right), \end{aligned}$$

$$\kappa^2(t) = \frac{\pi^2 t^2}{4a^2} \quad , \quad \tau(t) = \frac{\pi t}{2a}.$$

Thus, the problem of not being able to differentiate of the curvature at the point $t = 0$ is eliminated.

Example 4.14. *Let a non-unit speed space curve be*

$$\beta(t) = \left(t\cos t - \sin t, \cos t + t\sin t, \frac{t^2}{2}\right),$$

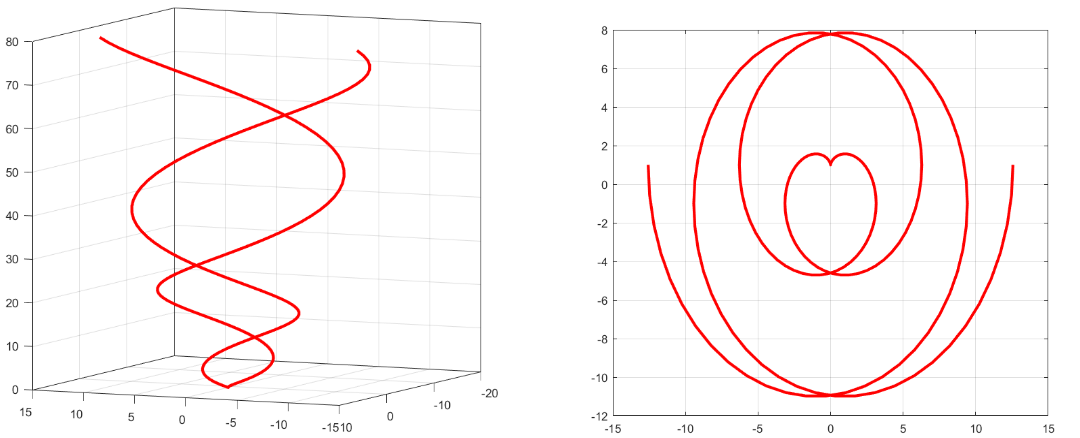

Figure 4: The curve $\alpha(t) = (t\cos t - \sin t, \cos t + t\sin t, t)$

Figure 4. Then the first, second and third derivative vectors of $\beta(t)$ are as follows:

$$\beta'(t) = (-t\sin t, t\cos t, t),$$

$$\beta''(t) = (-t\cos t - \sin t, \cos t - t\sin t, 1),$$

$$\beta'''(t) = (-2\cos t + t\sin t, -t\cos t - 2\sin t, 0),$$

Since $\left\|\beta'(t)\right\| = v(t) = \sqrt{2}\,|t|$, *the curve be* $\beta(t)$ *is not an unit speed curve. So the Frenet vectors, the curvature, the torsion, the Darboux vector, the unit vector in direction of the Darboux vector of the curve* $\beta(t)$ *are obtained as follows:*

$$\begin{aligned} T(t) &= \left(-\frac{t}{\sqrt{2}\,|t|}\sin t, \frac{t}{\sqrt{2}\,|t|}\cos t, \frac{t}{\sqrt{2}\,|t|}\right), \\ N(t) &= \left(\frac{t}{|t|}\cos t, \frac{t}{|t|}\sin t, 0\right), \\ B(t) &= \left(\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\right), \end{aligned}$$

$$\kappa(t) = \frac{1}{\sqrt{2}\,|t|} \quad , \quad \tau(t) = \frac{1}{2t},$$

$$W(t) = \left(\frac{t-t^3}{\sqrt{2}\,|t|}\sin t, \frac{t^3-t}{\sqrt{2}\,|t|}\cos t, \frac{t+t^3}{\sqrt{2}\,|t|}\right),$$

$$C(t) = \left(\frac{t-t^3}{\sqrt{2}\,|t|\,(1+t^4)}\sin t, \frac{t^3-t}{\sqrt{2}\,|t|\,(1+t^4)}\cos t, \frac{t+t^3}{\sqrt{2}\,|t|\,(1+t^4)}\right).$$

Now, let's examine the left and right limits of the vectors $T(t)$ *and* $N(t)$. *Since* $\lim_{t\to 0^+} T(t) \neq \lim_{t\to 0^-} T(t)$, $\lim_{t\to 0^+} N(t) \neq \lim_{t\to 0^-} N(t)$ *and* $\lim_{t\to 0^+} W(t) \neq \lim_{t\to 0^-} W(t)$, *there is no limit at* $t = 0$. *So the tangent vector and principal normal vector are discontinuous at* $t = 0$. *And, it is clear that, the curvature is not differentiable at* $t = 0$. *So, to solve the problem at* $t = 0$, *let's create the modified orthogonal frame of the curve. The modified orthogonal frame with the curvature* $\kappa(t)$ *of the non-unit speed curve* $\beta(t)$ *is gotten as follows:*

$$E_1(t) = (-t\sin t, t\cos t, t),$$

$$E_2(t) = \left(\sqrt{2}t\cos t, \sqrt{2}t\sin t, 0\right),$$

$$E_3(t) = \left(\sqrt{2}t^2\sin t, -\sqrt{2}t^2\cos t, \sqrt{2}t^2\right),$$

where,

$$\langle E_1(t), E_2(t)\rangle = \langle E_2(t), E_3(t)\rangle = \langle E_1(t), E_3(t)\rangle = 0,$$

$$\langle E_1(t), E_1(t)\rangle = \langle E_2(t), E_2(t)\rangle = 2t^2, \qquad \langle E_3(t), E_3(t)\rangle = 4t^4$$

and

$$E_1(t) \wedge E_2(t) = E_3(t),$$

$$E_2(t) \wedge E_3(t) = \left(-2t^3\sin t, 2t^3\cos t, 2t^3\right) = v^4(t)\kappa^2(t)E_1(t),$$

$$E_3(t) \wedge E_1(t) = \left(2\sqrt{2}t^3\cos t, 2\sqrt{2}t^3\sin t, 0\right) = v^2(t)E_2(t).$$

Also, the curvature, the torsion, the Darboux vector and the unit vector in direction of the Darboux vector of the modified orthogonal frame with the curvature $\kappa(t)$ *of the non-unit speed curve* $\beta(t)$ *are as following:*

$$\kappa^2(t) = \frac{1}{2t^2} \quad , \quad \tau(t) = \frac{1}{2t},$$

$$D(t) = \left(\frac{1-t^2}{\sqrt{2}}\sin t, \frac{t^2-1}{\sqrt{2}}\cos t, \frac{1+t^2}{\sqrt{2}}\right),$$

$$G(t)=\left(\frac{1-t^2}{\sqrt{2}(1+t^4)}\sin t,\frac{t^2-1}{\sqrt{2}(1+t^4)}\cos t,\frac{1+t^2}{\sqrt{2}(1+t^4)}\right).$$

Thus, instead of the Frenet frame of the curve, which causes problem at $t = 0$, *the properties of the curve at this point can be examined with this modified orthogonal frame.*

5. Conclusions

At singular points (or sharp points) on the analytical curves (or the discontinuous curves), one or more of the Frenet vectors, or the curvature and torsion functions, cannot be differentiated because their right and left limits are not the same. At these points, the use of the modified orthogonal frame instead of the Frenet frame is sufficient and necessary to solve the problem at that point. Apart from this, there is no harm in creating a modified orthogonal frame of any regular curve. Just like the Frenet frame, the characteristic features of the curve can also be examined with the modified orthogonal frame. Since the Frenet frame provides more ease of operation, the modified orthogonal frame is not preferred much in studies on regular curves. In this study, we have shown that the modified orthogonal frame works well not only for unit speed curves, but also for non-unit speed curves.

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