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A New Generalization of Szász-Kantorovich Operators on Weighted Space

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Abstract. The purpose of this article is to define a new generalization of Szász-Kantorovich operators. First, by using the Korovkin theorem on the new operator we define, its convergence properties and rates are examined. Then, the Voronovskaja-type theorem for the new operator is proven. Additionally, with the help of the modulus of continuity in the weighted space, rate of convergence the new operator is examined, and a theorem is proven for the operator we define by using functions that satisfy the Lipschitz condition. Finally, the convergence is demonstrated more clearly by numerical examples and plots.

1. INTRODUCTION

Linear positive operators have been studied by several mathematicians in the context of many fields of mathematics from the past to the present. In 1885, Weierstrass [18] proved the existence of a polynomial for any function in a finite interval that converges to this function within the same finite interval. However, he did not provide information on the properties of such a polynomial. In 1912, Russian mathematician Bernstein [3] defined the following operator as proof of the concept defined by Weierstrass.

$$B_b\left(j\left(i\right);s\right)=:B_b\left(j;s\right)=\sum_{l=0}^{b}j\left(\frac{l}{b}\right)\binom{b}{l}s^l\left(1-s\right)^{b-l} \tag{1}$$

here $j\in C[0,1]$, $s\in[0,1]$ and $b\in\mathbb{N}$.

After Bernstein's study, in different places and at different times, Bohman [4] in 1952 and Korovkin [11] in 1953 presented important theorems that proved the possibility of this convergence by providing only three conditions and pioneered this field regarding the convergence of positive operators to a function that is an element of $C[a,b]$ in a finite range. These theorems are generally known as the Korovkin conditions.

Afterward, studies in the field of convergence theory gained momentum, and several mathematicians [1, 2, 7, 9, 10, 12–14] conducted studies in this field.

L. V. Kantorovich [8] in 1930 and O. Szász [15] in 1950 completed their generalizations in which they defined the Bernstein operator in different spaces. These operators they defined are known by their names.

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Later, in 1985, the operator called the Szász-Kantorovich operator was studied by many mathematicians [6, 16]. The classical Szász-Kantorovich operator is as follows: Let $S_{b,f}(s) = e^{-bs}\frac{(bs)^f}{f!}$

$$W_u^j(s) = ue^{-us}\sum_{v=0}^{\infty}\left[\int_{\frac{v}{u}}^{\frac{v+1}{u}} j(s)\,ds\right]\frac{(us)^v}{v!}\;;\;\; u>0 \tag{2}$$

In this study, we defined an operator as a generalization of classical Szász-Kantorovich operators as follows:

$$Z_b(j\,;s) = \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{(f+1)}{b}\frac{(b+c)}{(b+e)}} j(i)\,di\,;0\le i<\infty \tag{3}$$

here $c,e\in\mathbb{R}$ and $0\le c\le e$.

Remark 1.1. *Szász-Kantorovich type operators defined by* (3) *are linear and positive.*

2. PRELIMINARIES

This section presents the examination of the convergence characteristics of our operator (3) which is a novel generalization of classical Szász Kantorovich operators (2). Additionally, for an arbitrary $A>0$, the uniform convergences of the operator are examined for continuous functions and functions bounded on the entire real axis in the closed compact interval $[0,A]$. The convergences and convergence rates are calculated in intervals diverging to $[0,\infty)$ and weighted spaces. The Voronovskaja-type theorem for functions that are differentiable in $[0,\infty)$ whose derivatives are in $C_p[0,\infty)$ is calculated.

To make these calculations and demonstrate that our operator satisfies the Korovkin conditions, let us calculate the $1, i, i^2, i^3$, and i^4 values of our operator. After this, with the help of these values, let us calculate its central moments.

Theorem 2.1. *The operator* (3) *satisfies the following equations for* $\forall s\in[0,A]$

$$Z_b(1\,;s) = 1$$

$$Z_b(i\,;s) = s + \frac{(c-e)}{(b+e)}s + \frac{1}{2}\frac{(b+c)}{b(b+e)}$$

$$Z_b\left(i^2\,;s\right) = s^2 + \frac{(2bc+c-2be-e)}{(b+e)^2}s^2 + 2\frac{(b+c)^2}{b(b+e)^2}s + \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}$$

$$\begin{aligned}Z_b\left(i^3\,;s\right) &= s^3 + \frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(e+b)^3}s^3\\ &\quad+\frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}s^2 + \frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\end{aligned}$$

$$\begin{aligned}Z_b\left(i^4\,;s\right) &= s^4 + \frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4}s^4\\ &\quad+8\frac{(b+c)^4}{b(b+e)^4}s^3 + 15\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 6\frac{(b+c)^4}{b^3(b+e)^4}s + \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\end{aligned}$$

Proof.

$$
\begin{aligned}
Z_b(1\,;s) &= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} 1 di \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \left(\frac{f+1}{b}\frac{(b+c)}{(b+e)} - \frac{f}{b}\frac{(b+c)}{(b+e)} \right) \\
&= \frac{b(b+e)}{(b+c)} . \frac{(b+c)}{b(b+e)} (f+1-f) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
Z_b(i\,;s) &= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i di \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{2} \frac{(b+c)^2}{b^2(b+e)^2} \left(\frac{(f+1)^2}{(b)^2}\frac{(b+c)^2}{(b+e)^2} - \frac{(f)^2}{(b)^2}\frac{(b+c)^2}{(b+e)^2} \right) \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{2} \frac{(b+c)^2}{b^2(b+e)^2} \left(f^2 + 2f + 1 - f^2 \right) \\
&= \frac{b(b+e)}{(b+c)} \frac{(b+c)^2}{b^2(b+e)^2} \frac{1}{2} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} (2f+1) \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=1}^{\infty} \frac{(bs)^{f-1}}{f(f-1)!} (bs) \frac{1}{2} 2f + \frac{(b+c)}{b(b+e)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{2} 1 \\
&= \frac{(b+c)}{b(b+e)} (bs) + \frac{1}{2} \frac{(b+c)}{b(b+e)} \\
&= \frac{(b+c)}{(b+e)} s + \frac{1}{2} \frac{(b+c)}{b(b+e)} \\
&= s + \frac{(c-e)}{(b+e)} s + \frac{1}{2} \frac{(b+c)}{b(b+e)}
\end{aligned}
$$

$$
\begin{aligned}
Z_b\left(i^2\,;s\right) &= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i^2 di \\
&= \frac{b(b+e)}{(b+c)} e^{-bs} \sum_{f=0}^{\infty} \frac{(bs)^f}{f!} \frac{1}{3} \left(\frac{(f+1)^3}{(b)^3}\frac{(b+c)^3}{(b+e)^3} - \frac{(f)^3}{(b)^3}\frac{(b+c)^3}{(b+e)^3} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{3}\frac{(b+c)^3}{b^3(b+e)^3}(f^3+3f^2+3f+1-f^3)\\
&= \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}(3f^2+3f+1) \quad ; f^2=f(f-1)+f\\
&= \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}3f^2\\
&\quad+\frac{1}{3}3\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}f+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}1\\
&= \frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}(f(f-1)+f)\\
&\quad+\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\
&= \frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2f(f-1)\\
&\quad+\frac{(b+c)^2}{b^2(b+e)^2}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f+\frac{(b+c)^2}{b^2(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\
&= \frac{(b+c)^2}{(b+e)^2}s^2+\frac{(b+c)^2}{b(b+e)^2}s+\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\
&= \frac{(b+c)^2}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\
&= s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}
\end{aligned}
$$

$$
\begin{aligned}
Z_b\left(i^3;s\right) &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i^3\,di\\
&= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}\left(\frac{(f+1)^4}{(b)^4}\frac{(b+c)^4}{(b+e)^4}-\frac{(f)^4}{(b)^4}\frac{(b+c)^4}{(b+e)^4}\right)\\
&= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}\frac{(b+c)^4}{b^4(b+e)^4}((f+1)^4-f^4)\\
&= \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}(f^4+4f^3+6f^2+4f+1-f^4)\\
&= \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}4f^3+\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}6f^2\\
&\quad+\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{4}4f+\frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}1
\end{aligned}
$$

If the equation $f^3 = f(f-1)(f-2) + 3f(f-1) + 3f - 2f$ is substituted into the last equation, the following is obtained:

$$\begin{aligned}
&= \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(f(f-1)(f-2)+3f(f-1)+3f-2f\right)\\
&+ \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{3}{2}\left(f(f-1)+f\right)\\
&+ \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\\
&= \frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!}(bs)^3 f(f-1)(f-2)\\
&+3\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1)\\
&+3\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f - 2\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f\\
&+\frac{3}{2}\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs)f\\
&+\frac{3}{2}\frac{(b+c)^3}{b^3(b+e)^3}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1)\\
&+\frac{(b+c)^3}{b^3(b+e)^3}s + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\\
&= \frac{(b+c)^3}{(b+e)^3}s^3 + \frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}s^2 + \frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\\
&= s^3 + \frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}s^3\\
&+\frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}s^2 + \frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s + \frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}
\end{aligned}$$

$$\begin{aligned}
Z_b\left(i^4;s\right) &= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} i^4\,di\\
&= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{5}\left(\frac{(f+1)^5}{(b)^5}\frac{(b+c)^5}{(b+e)^5} - \frac{(f)^5}{(b)^5}\frac{(b+c)^5}{(b+e)^5}\right)\\
&= \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\frac{1}{5}\frac{(b+c)^5}{b^5(b+e)^5}(f^5+5f^4+5f+10f^2+10f^3+1-f^5)
\end{aligned}$$

If the equations

$$f^3 = f(f-1)(f-2) + 3f(f-1) + 3f - 2f$$

and

$$f^4 = f(f-1)(f-2)(f-3) + 6f^3 + 11f^2 + 6f$$

are entered into the last equation, the following is obtained:

$$\begin{aligned} = {} & \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\, e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(5\left(f(f-1)(f-2)(f-3)+6f^3+11f^2+6f\right)\right) \\ & +10\,(f(f-1)(f-2)+3f(f-1)+3f-2f)+10(f(f-1)+f)+5f+1 \end{aligned}$$

Here, the sum can be calculated in two parts for easy operation:

$$\begin{aligned} = {} & \frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\, e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\left(5\left(f(f-1)(f-2)(f-3)+6f^3+11f^2+6f\right)\right) \\ & +10\,(f(f-1)(f-2)+3f(f-1)+3f-2f)+10(f(f-1)+f)+5f+1 \end{aligned}$$

$$\begin{aligned} I_1 = {} & \frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=4}^{\infty}\frac{(bs)^{f-4}}{f(f-1)(f-2)(f-3)(f-4)!} \\ & f(f-1)(f-2)(f-3)(bs)^4 \\ & +6\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!}f(f-1)(f-2)(bs)^3 \\ & +18\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)!}f(f-1)(bs)^2 \\ & +18\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) - 12\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) \\ & +11\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)!}f(f-1)(bs)^2 \\ & +11\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) + 6\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)!}f(bs) \\ = {} & \frac{(b+c)^4}{(b+e)^4}s^2 + 6\frac{(b+c)^4}{b(b+e)^4}s^3 + 18\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 18\frac{(b+c)^4}{b^3(b+e)^4}s \\ & -12\frac{(b+c)^4}{b^3(b+e)^4}s + 11\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 11\frac{(b+c)^4}{b^3(b+e)^4}s + 6\frac{(b+c)^4}{b^3(b+e)^4}s \\ = {} & \frac{(b+c)^4}{(b+e)^4}s^2 + 6\frac{(b+c)^4}{b(b+e)^4}s^3 + 29\frac{(b+c)^4}{b^2(b+e)^4}s^2 + 23\frac{(b+c)^4}{b^3(b+e)^4}s \end{aligned}$$

and

$$\begin{aligned}
I_2 &= \frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=3}^{\infty}\frac{(bs)^{f-3}}{f(f-1)(f-2)(f-3)!}(bs)^3 f(f-1)(f-2)\\
&+\frac{30}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1)\\
&+\frac{30}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f\\
&-\frac{20}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f\\
&+\frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=2}^{\infty}\frac{(bs)^{f-2}}{f(f-1)(f-2)!}(bs)^2 f(f-1)\\
&+\frac{10}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f\\
&+\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=1}^{\infty}\frac{(bs)^{f-1}}{f(f-1)!}(bs) f+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^{f}}{f!}\\
&= 2\frac{(b+c)^4}{b(b+e)^4}s^3+6\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s-4\frac{(b+c)^4}{b^3(b+e)^4}s\\
&\quad+2\frac{(b+c)^4}{b^2(b+e)^4}s^2+2\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
&= 2\frac{(b+c)^4}{b(b+e)^4}s^3+8\frac{(b+c)^4}{b^2(b+e)^4}s^2+5\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}
\end{aligned}$$

The following is obtained from I_1 and I_2

$$\begin{aligned}
&= \frac{(b+c)^4}{(b+e)^4}s^4+8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
Z_b\left(i^4\,;s\right) &= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4\\
&\quad+8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}x+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}
\end{aligned}$$

QED. □

Theorem 2.2. *Some of the central moments for our new Szász-Kantorovich operator* (3) *are as follows:*

$$Z_b\left((i-s)^0\,;s\right)=1$$

$$Z_b\left((i-s)\;;\;s\right)=\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}$$

$$Z_b\left((i-s)^2\ ;\ s\right) = \left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2 + \left(2\frac{(b+c)^2}{b\,(b+e)^2}-\frac{(b+c)}{b\,(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}$$

$$Z_b\left((i-s)^3\ ;\ s\right) = \left(\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3} -3\frac{(2bc+c-2be-e)}{(b+e)^2}+\ 3\frac{(c-e)}{(b+e)}\right)s^3 + \left(4\frac{(b+c)^3}{b\,(b+e)^3}-6\frac{(b+c)^2}{b\,(b+e)^2}+\frac{3}{2}\frac{(b+c)}{b\,(b+e)}\right)s^2 + \left(\frac{7}{2}\frac{(b+c)^3}{b^2\,(b+e)^3}-\frac{(b+c)^2}{b^2\,(b+e)^2}\right)s+\frac{1}{4}\frac{(b+c)^3}{b^3\,(b+e)^3}$$

$$Z_n\left((t-x)^4\ ;\ x\right) = \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4} -\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3} +6\frac{(2bc+c-2be-e)}{(b+e)^2}-\ 4\frac{(c-e)}{(b+e)}\right)s^4 + \left(8\frac{(b+c)^4}{b\,(b+e)^4}-16\frac{(b+c)^3}{b\,(b+e)^3}+12\frac{(b+c)^2}{b\,(b+e)^2}-2\frac{(b+c)}{b\,(b+e)}\right)s^3 + \left(15\frac{(b+c)^4}{b^2\,(b+e)^4}-14\frac{(b+c)^3}{b^2\,(b+e)^3}+2\frac{(b+c)^2}{b^2\,(b+e)^2}\right)s^2 + \left(6\frac{(b+c)^4}{b^3\,(b+e)^4}-\frac{(b+c)^3}{b^3\,(b+e)^3}\right)s\ +\frac{1}{5}\frac{(b+c)^4}{b^4\,(b+e)^4}$$

Proof.

$$Z_b\left((i-s)^0\,;s\right)=Z_b\,(1;s)=1$$

it is clear that.

$$\begin{aligned} Z_b\,((i-s)\ ;\ s) &= Z_b\,(i;s)-sZ_b\,(1;s) \\ &= s+\frac{(c-e)}{(b+e)}s+\frac{1}{2}\frac{(b+c)}{b\,(b+e)}-s.1 \\ &= \frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)} \end{aligned}$$

$$\begin{aligned}
Z_b\left((i-s)^2\ ;\ s\right) &= Z_b\left(i^2-2si+s^2\ ;s\right)\\
&= Z_b\left(i^2;s\right)-2sZ_b\left(i;s\right)+s^2Z_b\left(1;s\right)\\
&= s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b\,(b+e)^2}s\\
&\quad +\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}-2s\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)+s^2.1\\
&= \left(2+\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}-2\right)s^2\\
&\quad +\left(2\frac{(b+c)^2}{b\,(b+e)^2}-\frac{(b+c)}{b\,(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}\\
&= \left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\\
&\quad +\left(2\frac{(b+c)^2}{b\,(b+e)^2}-\frac{(b+c)}{b\,(b+e)}\right)s+\frac{1}{3}\ \frac{(b+c)^2}{b^2\,(b+e)^2}
\end{aligned}$$

$$\begin{aligned}
Z_b\left((i-s)^3\ ;\ s\right) &= Z_b\left(i^3-3si^2+3is^2-s^3;s\right)\\
&= Z_b\left(i^3;s\right)-3s\,Z_b\left(i^2;s\right)+3s^2Z_b\left(i;s\right)-s^3Z_b\left(1;s\right)\\
&= s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}s^3\\
&\quad +\frac{9}{2}\ \frac{(b+c)^3}{b\,(b+e)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2\,(b+e)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3\,(b+e)^3}\\
&\quad -3s\left(s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b\,(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}\right)\\
&\quad +3s^2\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)-s^3\\
&= s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}s^3\\
&\quad +\frac{9}{2}\ \frac{(b+c)^3}{b\,(b+e)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2\,(b+e)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3\,(b+e)^3}\\
&\quad -3s^3-3\frac{(2bc+c-2be-e)}{(b+e)^2}s^3-6\frac{(b+c)^2}{b\,(b+e)^2}s^2\\
&\quad -\frac{(b+c)^2}{b^2\,(b+e)^2}s+3s^3+3\frac{(c-e)}{(b+e)}s^3+\frac{3}{2}\frac{(b+c)}{b\,(b+e)}s^2-s^3\\
&= \left(\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(b+e)^3}\right.\\
&\quad \left.-3\frac{(2bc+c-2be-e)}{(b+e)^2}+\ 3\frac{(c-e)}{(b+e)}\right)\ s^3
\end{aligned}$$

$$+\left(\frac{9}{2}\frac{(b+c)^3}{b(b+e)^3}-6\frac{(b+c)^2}{b(b+e)^2}+\frac{3}{2}\frac{(b+c)}{b(b+e)}\right)s^2$$
$$+\left(\frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}-\frac{(b+c)^2}{b^2(b+e)^2}\right)s+\frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}$$

$$\begin{aligned}
Z_b\left((i-s)^4\ ;\ s\right) &= Z_b\left(i^4-4si^3+6s^2i^2-4s^3i+s^4\ ;s\right)\\
&= Z_b\left(i^4\ ;s\right)-4sZ_b\left(i^3\ ;s\right)+6s^2Z_b\left(i^2\ ;e\right)-4s^3Z_b\left(i\ ;s\right)+s^4Z_b\left(1\ ;s\right)\\
&= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4\\
&\quad +8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
&\quad -4s\left(s^3+\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}s^3\right.\\
&\quad \left.\frac{9}{2}\ \frac{(b+c)^3}{b(b+e)^3}s^2+\frac{7}{2}\frac{(b+c)^3}{b^2(b+e)^3}s+\frac{1}{4}\frac{(b+c)^3}{b^3(b+e)^3}\right)\\
&\quad +6s^2\left(s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right)\\
&\quad -4s^3\left(s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right)+s^4\\
&= s^4+\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(n+b)^4}s^4\\
&\quad +8\frac{(b+c)^4}{b(b+e)^4}s^3+15\frac{(b+c)^4}{b^2(b+e)^4}s^2+6\frac{(b+c)^4}{b^3(b+e)^4}s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\\
&\quad -4s^4-4\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}s^4-18\frac{(b+c)^3}{b(b+e)^3}s^3\\
&\quad -14\frac{(b+c)^3}{b^2(b+e)^3}s^2-\frac{(b+c)^3}{b^3(b+e)^3}s+6s^4+6\frac{(2bc+c-2be-e)}{(b+e)^2}s^4\\
&\quad +12\frac{(b+c)^2}{b(b+e)^2}s^3+2\frac{(b+c)^2}{b^2(b+e)^2}s^2-4s^4-4\frac{(c-e)}{(b+e)}s^4-2\frac{(b+c)}{b(b+e)}s^3+s^4\\
&= \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4}\right.\\
&\quad -\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}\\
&\quad \left.+6\frac{(2bc+c-2be-e)}{(b+e)^2}-\ 4\frac{(c-e)}{(b+e)}\right)s^4\\
&\quad +\left(8\frac{(b+c)^4}{b(b+e)^4}-16\frac{(b+c)^3}{b(b+e)^3}+12\frac{(b+c)^2}{b(b+e)^2}-2\frac{(b+c)}{b(b+e)}\right)s^3
\end{aligned}$$

$$+\left(15\frac{(b+c)^4}{b^2(b+e)^4}-14\frac{(b+c)^3}{b^2(b+e)^3}+2\frac{(b+c)^2}{b^2(b+e)^2}\right)s^2$$
$$+\left(6\frac{(b+c)^4}{b^3(b+e)^4}-\frac{(b+c)^3}{b^3(b+e)^3}\right)s \;+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}$$

QED. □

Theorem 2.3. *Let $A > 0$, and $j \in C[0, A]$ be bounded on the entire real axis. In this case, the following is obtained:*

$$\lim_{b\to\infty}\left\|Z_b j - j\right\|_{C[0,A]} = 0$$

Proof. If we use the Korovkin theorem, for $b \to \infty$, it is sufficient to demonstrate the following:

i) $Z_b(1\,;s) \rightrightarrows 1$

ii) $Z_b(i\,;s) \rightrightarrows s$

iii) $Z_b\left(i^2\,;s\right) \rightrightarrows s^2$

The following is clear, $\|Z_b(1\,;s) - 1\|_{C[0,A]} = 0$. Thus,$Z_b(1\,;s) \rightrightarrows 1$ is obtained.

$$\begin{aligned}\|Z_b(i\,;s)-s\|_{C[0,A]} &= \max_{0\le s\le A}|Z_n(i\,;s)-s| \\ &= \max_{0\le s\le A}\left|s+\frac{(c-e)}{(b+e)}s+\frac{1}{2b}\frac{(b+c)}{(b+e)}-s\right| \\ &= \left|\frac{(c-e)}{(b+e)}A+\frac{1}{2b}\frac{(b+c)}{(b+e)}\right| \; buradan \\ &= \left|\frac{2b(c-e)}{2b(b+e)}A+\frac{(b+c)}{2b(b+e)}\right|\end{aligned}$$

We obtain $\lim_{b\to\infty}\left|\frac{2b(c-e)}{2b(b+e)}A+\frac{(b+c)}{2b(b+e)}\right| \to 0$, which shows that $Z_b(i\,;s) \rightrightarrows s$.

$$\begin{aligned}\left\|Z_b\left(i^2\,;s\right)-s^2\right\|_{C[0,A]} &= \max_{0\le s\le A}\left|Z_b\left(i^2\,;s\right)-s^2\right| \\ &= \max_{0\le s\le A}\left|s^2+\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}-s^2\right| \\ &= \max_{0\le s\le A}\left|\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right| \\ &\le \left|\frac{(2bc+c-2be-e)}{(b+e)^2}A^2+2\frac{(b+c)^2}{b(b+e)^2}A+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right|\end{aligned}$$

Since it is $\lim_{n\to\infty}\left|\frac{(2bc+c-2be-e)}{(b+e)^2}A^2+\frac{(b+c)^2}{(b+e)^2}\left(\frac{2}{b}A+\frac{1}{3}\frac{1}{b^2}\right)\right| \to 0$, $Z_b\left(i^2\,;s\right) \rightrightarrows s^2$ is obtained. QED. □

Let us now examine convergence in weighted spaces.

Theorem 2.4. *If $j \in C^0_\rho[0,\infty)$, then*

$$\lim_{b\to\infty}\left\|Z_b j - j\right\|_{\rho,[0,\infty)} = 0.$$

Proof. Because

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} = \sup_{s\in[0,\infty)} \frac{\left|Z_b\left(j\,;s\right) - j\left(s\right)\right|}{1+s^2}$$

and based on the properties of the modulus of continuity in weighted spaces, we obtain the following:

$$\lim_{b\to\infty} \|Z_b 1 - 1\|_{\rho,[0,\infty)} = \lim_{b\to\infty} \sup_{s\in[0,\infty)} \frac{|Z_b\left(1\,;s\right) - 1|}{1+s^2} = 0$$

Using the results we obtained in the central moments and the norm definition in weighted spaces, the following can be expressed

$$\begin{aligned}
\|Z_b i - s\|_{\rho,[0,\infty)} &= \sup_{s\in[0,\infty)} \frac{\left|s + \frac{(c-e)}{(b+e)}s + \frac{1}{2b}\frac{(b+c)}{(b+e)} - s\right|}{1+x^2} \\
&\le \frac{2b\left(c-e\right)}{2b\left(b+e\right)} + \frac{\left(b+c\right)}{2b\left(b+e\right)} \\
\lim_{b\to\infty} \|Z_b i - s\|_{\rho,[0,\infty)} &\le \lim_{b\to\infty}\left(\frac{2b\left(c-e\right)}{2b\left(b+e\right)} + \frac{\left(b+c\right)}{2b\left(b+e\right)}\right) \to \infty.
\end{aligned}$$

Likewise, because

$$\begin{aligned}
\left\|Z_b i^2 - s^2\right\|_{\rho,[0,\infty)} &= \sup_{s\in[0,\infty)} \frac{\left|s^2 + \frac{(2bc+c-2be-e)}{(b+e)^2}s^2 + 2\frac{(b+c)^2}{b(b+e)^2}s + \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2} - s^2\right|}{1+s^2} \\
&= \sup_{s\in[0,\infty)} \frac{\left|\frac{3b^2(2bc+c-2be-e)+6b(b+c)^2+(b+c)^2}{3b^2(b+e)^2}\right|}{1+s^2}
\end{aligned}$$

If $0 \le \frac{s^2}{1+s^2} \le 1$ based on the above, the following is obtained:

$$\begin{aligned}
&\le \frac{3b^2\left(2bc+c-2be-e\right) + 6b\left(b+c\right)^2 + \left(b+c\right)^2}{3b^2\left(b+e\right)^2} \\
\lim_{b\to\infty}\left\|Z_b i^2 - s^2\right\|_{\rho,[0,\infty)} &= \lim_{b\to\infty}\left(\frac{3b^2\left(2bc+c-2be-e\right) + 6b\left(b+c\right)^2 + \left(b+c\right)^2}{3b^2\left(b+e\right)^2}\right) \to \infty
\end{aligned}$$

Therefore,

$$\lim_{b\to\infty}\left\|Z_b j - j\right\|_\rho = 0$$

QED. □

Let as examine rate of convergence the operator in weighted spaces.

Theorem 2.5. *If* $j \in C_\rho^0\left[0,\infty\right)$ *, then*

$$\left\|Z_b j - j\right\|_{\rho,[0,\infty)} \le M\Omega\left(j\,;\sqrt{\frac{1}{b}}\right).$$

Here, $M = 808$*.*

Proof. Because our operator is linear and monotone, the following can be written:

$$\left|Z_b\left(j\,;s\right) - j\left(s\right)\right| \le Z_b\left(\left|j\left(i\right) - j\left(s\right)\right|\,;s\right)$$

Using the properties of the modulus of continuity in weighted spaces, the following is obtained:

$$\begin{aligned}\left|j(i)-j(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\Omega\left(j\,;\eta_b\right)\\ \left|j(i)-j(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\Omega\left(j\,;\eta_b\right).S_b\left(i\,;s\right)\\ S_b\left(i\,;s\right) &= \left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\end{aligned}$$

From here,

$$S_b\left(i\,;s\right) \leq \left\{\begin{array}{ll} 2\left(1+\eta_b^2\right) & |i-s| < \eta_b \\ 2\left(1+\eta_b^2\right)\frac{|i-s|^4}{\eta_b^4} & ;\ |i-s| \geq \eta_b \end{array}\right\}$$

and thus, the following can be written:

$$S_b\left(i\,;s\right) \leq 2\left(1+\eta_b^2\right)\left[1+\frac{|i-s|^4}{\eta_b^4}\right]$$

Based on the above, the following is obtained:

$$\begin{aligned}\left|Z_b\left(j\,;s\right)-j(s)\right| &\leq Z_b\left(\left|j(i)-j(s)\right|\,;s\right)\\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\Omega\left(j\,;\eta_b\right)Z_b\left(S_b\left(i\,;s\right);s\right)\\ &\leq 4\left(1+\eta_b^2\right)\Omega\left(j\,;\eta_b\right)\left(1+s^2\right)\left[1+\frac{1}{\eta_b^4}Z_b\left(i-s\right)^4;s\right]\\ &\leq \left(\frac{\left(4na^3+4n^3a+6n^2a^2+a^4-4nb^3-4n^3b+6n^2b^2+b^4\right)}{(n+b)^4}\right.\\ &\quad \left.+6\frac{(2na+a-2nb-b)}{(n+b)^2}\right)s^4+\left(8\frac{(n+a)^4}{n\left(n+b\right)^4}+12\frac{(n+a)^2}{n\left(n+b\right)^2}\right)s^3\\ &\quad +\left(37\frac{(n+a)^4}{n^2\left(n+b\right)^4}+2\frac{(n+a)^2}{n^2\left(n+b\right)^2}\right)s^2\\ &\quad +\left(28\frac{(n+a)^4}{n^3\left(n+b\right)^4}\right)x+\left(\frac{(n+a)^4}{5n^4\left(n+b\right)^4}\right)\end{aligned}$$

Consequently, we get

$$\begin{aligned}Z_b\left((i-s)^4;s\right) &\leq \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)+6\left(b+e\right)^2\left(2bc+c-2be-e\right)}{(n+b)^4}\right)s^4\\ &+ \left(\frac{8\left(b+c\right)^4+12\left(b+e\right)^2\left(b+c\right)^2}{b\left(b+e\right)^4}\right)s^3+\left(\frac{15\left(b+c\right)^4+2\left(b+e\right)^2\left(b+c\right)^2}{b^2\left(b+e\right)^4}\right)s^2\\ &+ \left(\frac{6\left(b+c\right)^4}{b^3\left(b+e\right)^4}\right)s+\left(\frac{(b+c)^4}{5b^4\left(b+e\right)^4}\right)\end{aligned}$$

If the suprema of both sides in $[0,\infty)$ are taken, and if the negative operations are removed,

$$\begin{aligned}\sup_{s\in[0,\infty)} Z_b\left((i-s)^4;s\right) &\leq \frac{\left(5b^4\left(4bc^3+4b^3c+6b^2c^2+c^4+6b^2e^2+e^4\right)+30b^4(b+e)^2(2bc+c)\right)}{5b^4(b+e)^4}\\ &+ \frac{\left(5b^4\left(40c^3(b+c)^4+60(b+e)^2(b+c)^2+185b^2(b+c)^4\right)\right)}{5b^4(b+e)^4}\\ &+ \frac{\left(5b^4\left(^2(b+e)^2(b+c)^2+140b(b+c)^4+(b+c)^4\right)\right)}{5b^4(b+e)^4}\end{aligned}$$

Because $c\leq e$ if c is replaced with e, the following is found:

$$\leq\ 100\frac{b^3}{(b+e)^4}\leq 100\frac{1}{b}$$

Hence,

$$\sup_{s\in[0,\infty)}\frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2}\leq 4\left(1+\eta_b^2\right)\Omega(j;\eta_b)\left[1+\frac{100}{\eta_b^4}\frac{1}{b}\right]$$

Is obtained. If $\eta_b=\sqrt{\frac{1}{b}}$ and because $\eta_b\to 0$,$\eta_b<1$ is found after a certain b. Therefore, when $M=808$, the following is obtained:

$$\left\|Z_b j-j\right\|_{\rho,[0,\infty)}\leq M\Omega\left(j;\sqrt{\frac{1}{b}}\right)$$

QED. □

Theorem 2.6. *Let j be a function that is differentiable in $[0,\infty)$ and $j'\in C_\rho^0[0,\infty)$, the following inequality is provided:*

$$\left\|Z_b j-j\right\|_{\rho,[0,\infty)}\leq L\sqrt{\frac{2}{b}}\Omega\left(j';\sqrt{\frac{2}{b}}\right).$$

Proof. Because j is a function that is differentiable in $[0,\infty)$ and $j'\in C_\rho^0[0,\infty)$, based on the mean value theorem, there is a u between i and s such that

$$j'(u)=\frac{j(i)-j(s)}{i-s}$$

As the equation will not change if we add $-j'(s)+j'(s)$ to the left-hand side, the following can be written:

$$j(i)-j(s)=(i-s)j'(u)+(i-s)\left[j'(u)-j'(s)\right] \tag{1}$$

When $|u-s|\leq|i-s|$, $\Omega(j;|u-s|)\leq\Omega(j;|i-s|)$ Therefore, the following inequality is true:

$$\begin{aligned}\left|j'(u)-j'(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|u-s|}{\eta_b}\right)\left(1+(u-s)^2\right)\Omega(j';\eta_b)\\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(1+\frac{|i-s|}{\eta_b}\right)\left(1+(i-s)^2\right)\Omega(j';\eta_b)\\ &= 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left[1+\frac{|i-s|}{\eta_b}\right.\\ &\quad\left.+(i-s)^2+\frac{|i-s|(i-s)^2}{\eta_b}\right]\Omega(j';\eta_b)\end{aligned}$$

Let us calculate the multiplication below in equation (1).

The following result is obtained:

$$\begin{aligned}|i-s|\ \left|j'(u)-j'(s)\right| &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)[|i-s| \\ &+\frac{(i-s)^2}{\eta_b}+|i-s|(i-s)^2+\frac{(i-s)^4}{\eta_b}\Bigg]\Omega\left(j';\eta_b\right)\end{aligned} \tag{2}$$

If we apply the operator to equation (1),

$$Z_b(j;s)-j(s)=Z_b((i-s);s)\,j'(s)+Z_b((i-s)[j'(u)-j'(s)];s)$$

From here, the following is obtained:

$$\left|Z_b(j;s)-j(s)\right|\leq Z_b(|i-s|;s)\left|j'(s)\right|+Z_b(|i-s|[j'(u)-j'(s)];s)$$

Here, let $Z_b(|i-s|;s)\left|j'(s)\right|=I_1$ and $Z_b(|i-s|[j'(u)-j'(s)];s)=I_2$.

Then,

$$\left|Z_b(j;s)-j(s)\right|\leq I_1+I_2$$

Using the Cauchy-Schwarz inequality, the following can be written:

$$I_1=Z_b(|i-s|;s)\left|j'(s)\right|\leq\sqrt{Z_b\left((i-s)^2;s\right)}\left|j'(s)\right|\leq\sqrt{A_b(b)}M_{j'}\left(1+s^2\right)$$

Here, $M_{j'}$ is a constant dependent on $j'-ne$ and

$$\begin{aligned}A_b(s) &= \left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\right. \\ &\left.+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+e)}{b(b+e)}\right)x+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\right).\end{aligned}$$

From (2), the following can be written:

$$\begin{aligned}I_2 &= Z_b\left(|i-s|\ \left|f'(u)-j'(s)\right|;s\right) \\ &\leq 2\left(1+\eta_b^2\right)\left(1+s^2\right)\left(\sqrt{Z_b\left((i-s)^2;s\right)}+\frac{1}{\eta_b}Z_b\left((i-s)^2;s\right)\right. \\ &\left.+\sqrt{Z_b\left((i-s)^2;s\right)}\sqrt{Z_b\left((i-s)^4;s\right)}+\frac{1}{\eta_b}Z_b((i-s)4;s)\right)\Omega\left(j';\eta_b\right)\end{aligned}$$

Here, let $Z_b\left((i-s)^4;s\right)=B_b(s)$; then consequently,

$$\begin{aligned}B_b(s) &= \left(\frac{\left(4bc^3+4b^3c+6b^2c^2+c^4-4be^3-4b^3e+6b^2e^2+e^4\right)}{(b+e)^4}\right.\\&-\frac{\left(3bc^2+3b^2c+c^3-3be^2-3b^2e-e^3\right)}{(n+b)^3}\\&\left.+6\frac{(2bc+c-2be-e)}{(b+e)^2}-4\frac{(c-e)}{(b+e)}\right)s^4\\&+\left(8\frac{(b+c)^4}{b(b+e)^4}-16\frac{(b+c)^3}{b(b+e)^3}+12\frac{(b+c)^2}{b(b+e)^2}-2\frac{(b+c)}{b(b+e)}\right)s^3\\&+\left(15\frac{(b+c)^4}{b^2(b+e)^4}-14\frac{(b+c)^3}{b^2(b+e)^3}+2\frac{(b+c)^2}{b^2(b+e)^2}\right)s^2\\&+\left(6\frac{(b+c)^4}{b^3(b+e)^4}-\frac{(b+c)^3}{b^3(b+e)^3}\right)s+\frac{1}{5}\frac{(b+c)^4}{b^4(b+e)^4}\end{aligned}$$

because

$$\begin{aligned}&I_1+I_2\le\\&\sqrt{A_b(s)}M_{j'}\left(1+s^2\right)+2\left(1+\eta_b^2\right)\left(1+s^2\right)\left[\sqrt{A_b(s)}+\frac{1}{\eta_b}A_b(s)+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\right]\Omega\left(j';\eta_b\right)\end{aligned}$$

The following is written:

$$\begin{aligned}\frac{\left|Z_b(j;s)-j(s)\right|}{1+s^2} &\le \sqrt{A_b(s)}M_{j'}+2\left(1+\eta_b^2\right)\left[\sqrt{A_b(s)}+\frac{1}{\eta_b}A_b(s)\right.\\&\left.+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\right]\Omega\left(j';\eta_b\right)\end{aligned}$$

If the supremum of each side in $[0,\infty)$ is taken, the following inequality is obtained:

$$\begin{aligned}\sup_{s\in[0,\infty)}\frac{\left|Z_b(j;x)-j(s)\right|}{1+s^2} &\le \sup_{s\in[0,\infty)}\sqrt{A_b(s)}M_{j'}+2\left(1+\eta_b^2\right)\left[\sqrt{A_b(s)}\right.\\&\left.+\frac{1}{\eta_b}A_b(s)+\sqrt{A_b(s)}\sqrt{B_b(s)}+\frac{1}{\eta_b}B_b(s)\right]\Omega\left(j';\eta_b\right)\end{aligned}$$

Here, we obtain

$$\begin{aligned}\sup_{s\in[0,\infty)}A_b(s) &= \sup_{s\in[0,\infty)}\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2\\&+\left(2\frac{(b+c)^2}{b(b+e)^2}-\frac{(b+c)}{b(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\&\le \frac{(2bc+c-2be-e)}{(b+e)^2}+2\frac{(b+c)^2}{b(b+e)^2}+\frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2}\\&\le \frac{6b(b+e)^2+(b+e)^2}{3b^2(b+e)^2}\end{aligned}$$

and the following result is found:

$$\sup_{s\in[0,\infty)} A_b(s) \leq \frac{2}{b}$$

From the result we obtained for Theorem 2.5 $\sup_{s\in[0,\infty)} B_b(s) \leq 100\frac{1}{b}$ is written. Hence,

$$\begin{aligned}\sup_{s\in[0,\infty)} \frac{\left|Z_b(j;s) - j(s)\right|}{1+s^2} &\leq \left(\sqrt{\frac{2}{b}}M_{j'} + 2\left(1+\eta_b^2\right)\left[\sqrt{\frac{2}{b}} + \frac{1}{\eta_b}\frac{2}{b} + \sqrt{\frac{2}{b}}\sqrt{100\frac{1}{b}} + \frac{1}{\eta_b}100\frac{1}{b}\right]\Omega(j';\eta_b)\right) \\ &= \sqrt{\frac{2}{b}}\left(M_{j'} + 2\left(1+\eta_b^2\right)\left[1 + \frac{1}{\eta_b}\sqrt{\frac{2}{b}} + \frac{10}{b} + \frac{1}{\eta_b}100\frac{1}{\sqrt{b}}\right]\Omega(j';\eta_b)\right)\end{aligned}$$

Here, if $\eta_b = \sqrt{\frac{2}{b}}$ is selected, the following is obtained:

$$\begin{aligned}\sup_{s\in[0,\infty)} \frac{\left|Z_b(j;s) - j(s)\right|}{1+s^2} &\leq \sqrt{\frac{2}{b}}\left(M_{j'} + 4\left[1+1+10+100\right]\Omega(j';\eta_b)\right) \\ &\leq L\sqrt{\frac{2}{b}}\Omega\left(j';\sqrt{\frac{2}{b}}\right)\end{aligned}$$

Here $L = M_{j'} + 448$. QED. □

Theorem 2.7. *If j is a function providing the Lipschitz condition and $0 < \alpha \leq 1$, the following equation is true:*

$$\left\|Z_b(j;s) - j(s)\right\|_{C[0,A]} = 0\left(\left(\frac{4}{b}\right)^{\alpha/2}\right).$$

Proof. When $Z_b(1;s) = 1$, and because the operator is linear, the following can be written:

$$\begin{aligned}\left|Z_b(j;s) - j(s)\right| &= \left|Z_b(j;s) - j(s)Z_b(1;s)\right| \\ &= \left|Z_b(j;s) - Z_b(j(s);s)\right| \\ \left|Z_b(j;s) - j(s)\right| &\leq \left(Z_b\left|j(i) - j(s)\right|;s\right).\end{aligned}$$

Because j satisfies the Lipschitz condition and $\left|j(i) - j(s)\right| \leq M|i-s|^\alpha$,the following result is obtained:

$$\begin{aligned}\left|Z_b(j;s) - j(s)\right| &\leq \frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} M|i-s|^\alpha\, di \\ &\leq M\frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} |i-s|^\alpha\, di \\ &= M\frac{b(b+e)}{(b+c)}e^{-bs}\sum_{f=0}^{\infty}\frac{(bs)^f}{f!}\int_{\frac{f}{b}\frac{(b+c)}{(b+e)}}^{\frac{f+1}{b}\frac{(b+c)}{(b+e)}} |i-s|^\alpha\, di \\ &= MZ_b\left(|i-s|^\alpha;s\right)\end{aligned}$$

From Hölder's inequality, we obtain the following:

$$Z_b\left(|i-s|^{\alpha};s\right) \leq Z_b\left((i-s)^2;s\right)^{\alpha/2}$$

Therefore,

$$Z_b\left(|i-s|^{\alpha};s\right) \leq M\left(\left(\left(\frac{(2bc+c-2be-e)}{(b+e)^2}-2\frac{(c-e)}{(b+e)}\right)s^2+\left(2\frac{(b+c)^2}{b\,(b+e)^2}-\frac{(b+c)}{b\,(b+e)}\right)s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}\right)\right)^{\alpha/2}$$

If the maximum of the inside expression is taken,

$$\max_{0\leq s\leq A}\frac{(2bc+c-2be-e)}{(b+e)^2}s^2+2\frac{(b+c)^2}{b\,(b+e)^2}s+\frac{1}{3}\frac{(b+c)^2}{b^2\,(b+e)^2}$$

$$\begin{aligned}
&= \left(\frac{3b^2\,(2bc+c-2be-e)}{3b^2\,(nb+e)^2}+\frac{6b\,(b+c)^2}{3b^2\,(b+e)^2}+\frac{(b+c)^2}{3b^2\,(b+e)^2}\right)\\
&= \frac{3b^2\,(2bc+c-2be-e)+6b\,(b+c)^2+(b+c)^2}{3b^2\,(b+e)^2}
\end{aligned}$$

is obtained. If $a=b$ is selected here,

$$\begin{aligned}
&\leq \frac{(b+e)^2\,(6b+1)}{3b^2\,(b+e)^2}\\
&\leq \frac{6\,(b+1)}{3b^2}\\
&\leq \frac{2\,(b+1)}{b^2}\\
&\leq 2\left(\frac{1}{b}+\frac{1}{b^2}\right)\\
&\leq \frac{4}{b}
\end{aligned}$$

Then,

$$\left|Z_b\,(j\,;s)-j\,(s)\right| \leq M\left(\frac{4}{b}\right)^{\alpha/2}$$

and thus,

$$\left\|Z_b\,(j\,;s)-j\,(s)\right\|_{C[0,A]} \leq M\left(\frac{4}{b}\right)^{\alpha/2}$$

QED. □

Theorem 2.8. *Let the functions* $j\in[0,A]$ *and* j, j', j'' *be bounded functions in* $[0,A]$, *the following limit*

$$\lim_{b\to\infty}(b+e)\left(Z_b\,(j\,;s)-j\,(s)\right)=(c+1-e)\,j'\,(c)+xj''\,(s)\,.$$

Proof. The Taylor series expansion of the function j at point s and the form of this expansion in the operator are as follows:

$$j\,(i)=j\,(s)+\frac{1}{1!}j'\,(s)\,(i-s)+\frac{1}{2!}j''\,(s)\,(i-s)^2+\frac{1}{3!}j'''\,(s)\,(i-s)^3+\frac{1}{4!}j^4\,(s)\,(i-s)^4+...$$

$$j(i) - j(s) = \frac{1}{1!} j'(s)(i-s) + \frac{1}{2!} j''(s)(i-s)^2 + (i-s)^2 \mu(i-s)$$

$$\mu(i-s) = \left(\frac{1}{3!} j'''(s)(i-s)^3 + \frac{1}{4!} j^4(s)(i-s)^4 + ...\right)$$

$$Z_b(j;s) - j(s) = Z_b((i-s)\ ;s)\, j'(s) + \frac{1}{2} Z_b\left((i-s)^2\ ;s\right) j''(s) + Z_b\left((i-s)^2 \mu(i-s);s\right)$$

If we substitute in the central moments in the last equation, we obtain the following:

$$\begin{aligned} Z_b(j;s) - j(s) &= \left(\frac{(c-e)}{(b+e)} s + \frac{1}{2b} \frac{(b+c)}{(b+e)}\right) j'(s) \\ &+ \frac{1}{2}\left(\left(\frac{(2bc + c - 2be - e)}{(b+e)^2} - 2\frac{(c-e)}{(b+e)}\right) s^2 \right. \\ &+ \left(2\frac{(b+c)^2}{b(b+e)^2} - \frac{(b+c)}{b(b+n)}\right) s \\ &\left. + \frac{1}{3} \frac{(b+c)^2}{b^2(b+e)^2}\quad j''(s) + Z_b\left((i-s)^2 \mu(i-s);s\right)\right) \end{aligned}$$

If both sides of the equation are multiplied by $(b+e)$, we obtain the following result:

$$\begin{aligned} (b+e)(Z_b(j;s) - j(s)) &= (b+e)\left(\frac{(c-e)}{(b+e)} s + \frac{1}{2b}\frac{(b+c)}{(b+e)}\right) j'(s) + \frac{1}{2}(b+e) \\ &\left(\left(\frac{(2bc + c - 2be - e)}{(b+e)^2} - 2\frac{(c-e)}{(b+e)}\right) s^2 \right. \\ &+ \left(2\frac{(b+c)^2}{b(b+e)^2} - \frac{(b+c)}{b(b+n)}\right) s \\ &\left. + \frac{1}{3}\frac{(b+c)^2}{b^2(b+e)^2} j''(s) + (b+e) Z_b\left((i-s)^2 \mu(i-s);s\right)\right) \end{aligned}$$

Because $\lim_{i \to s} \mu(i - xi) = 0$ is bounded, and thus, the following equation can be written:

$$(b+e) Z_b\left((i-s)^2 \mu(i-s);s\right) \le \sqrt{(b+e) Z_b\left((i-s)^4;s\right)} \sqrt{(b+e) Z_b\left(\mu(i-s)^2;s\right)}$$

As $\lim_{b\to\infty} (b+e) Z_b\left((i-s)^4;s\right) = 0$ when the limit of the last equation is taken, we obtain the following:

$$\lim_{b\to\infty} (b+e)(Z_b(j;s) - j(s)) = (c+1-e) j'(s) + s j''(s)$$

QED. □

3. MAIN RESULTS

This section will review the main results. That is, we will exemplify the results of our operator's convergence with graphs and numerical values.

Let us now present the plots that we drew on the Maple program showing the convergence of our operator $Z_b(j;s)$ to the function $h(s) = \sin\left(\frac{\pi s}{2}\right)\sqrt{s}$ for different b values.

Figure 1: Convergence of the operators S and Z to the function $h(s)$ for $b = 10$

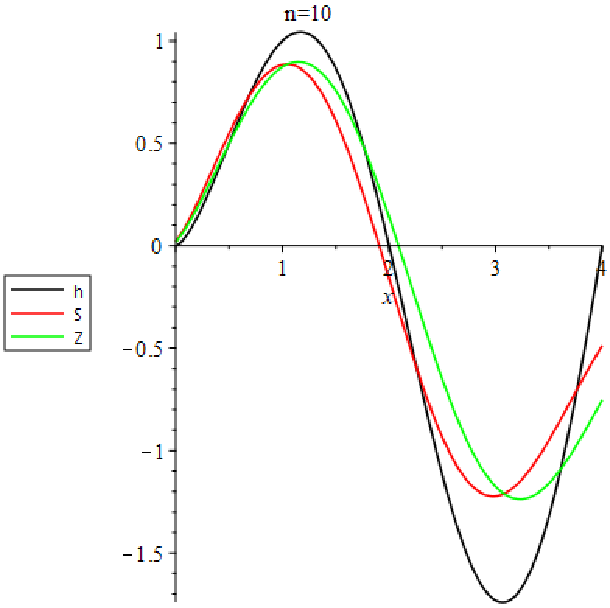


Figure 2: Convergence of the operators S and Z to the function $h(s)$ for $b = 20$

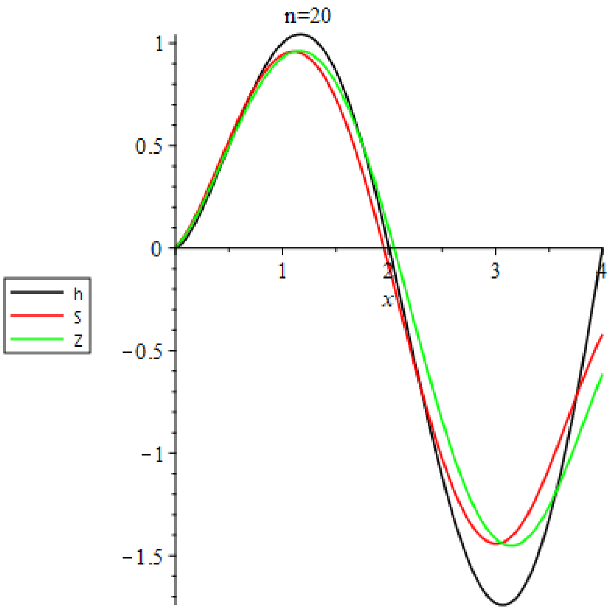
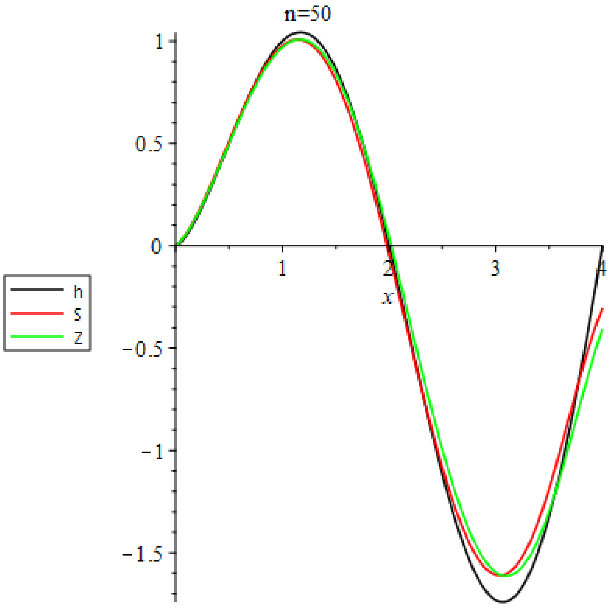


Figure 3: Convergence of the operators S and Z to the function $h(s)$ for $b = 50$



Then, we also demonstrate the convergence numerically with the numerical value table including the error margins. In the plots and the table, S refers to the classical Szász operator, while Z is the operator that we defined.

As seen here, as the values of b increase, the convergence becomes clearer, and our operator Z shows a better convergence than the operator S. Now, let us express the convergence rats of the operators S and Z with numerical values.

When

$$N(s) = \left| \frac{W_u^j(h;s) - h(s)}{Z_b(h;s) - h(s)} \right|$$

for different b and s values in $N(s)$, the following table of numerical values can be given.

$b - s$	$0,1$	$1,5$	$2,5$	4
10	1,02755	0,99899	1,284817	0,92587
100	1,01513	0,96156	1,25418	0,97556
300	1,01415	0,95843	1,25139	0,98571
500	1,01397	0,9578	1,25085	0,98889
700	1,01392	0,95752	1,25072	0,99059
1000	1,01403	0,95732	1,25061	0,99208

Table 1: $N(s)$ rate results for different b and s values.

The values given in the table show better convergence by S when they are greater than 1 and better convergence by Z when they are smaller than 1. Because the values are approximately 1 in general, it may be stated that these to operators are indeed equivalent.

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