TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 2, 107–115
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

On the Stability of Finite Difference Scheme for the Schrödinger Equation Including Momentum Operator

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Abstract. In this paper, we apply the finite difference method to a Schrödinger equation which contains a momentum operator. For this, we constitute a difference scheme. A priori estimate for the solution of difference scheme is obtained. By using this estimate, we prove that the difference scheme is unconditionally stable.

1. INTRODUCTION

Schrödinger equation,

$$i\hbar\frac{\partial u}{\partial t}(\varsigma,t)=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\varsigma,t)\right]u(\varsigma,t)=(T+V)\,u(\varsigma,t)$$

is a partial differential equation, where $i^2=-1$, ς and t are the variables of space and time, respectively, $u(\varsigma,t)$ is a wave function; $\hbar=\frac{h}{2m}$ is the reduced Planck's constant; h is the Planck's constant; m is the mass of particle; $T=\frac{p^2}{2m}$ is the kinetic energy operator; $p=-i\hbar\nabla$ is the momentum operator; $V=V(\varsigma,t)$ is the potential energy operator; ∇ is the gradient operator; ∇^2 is the Laplace operator.

As seen, the left hand side (l.h.s.) of above-mentioned equation describes the ratio of change of wave function u according to time, namely; Schrödinger equation is a equation describing how the energy of a quantum mechanical system evolves in time. It is a very sophisticated model applicable to many disciplines in engineering and applied sciences.

Many researchers analyzed the solutions of different versions of Schrödinger equation by using various methods (exactly, approximately or numerically). For example, Khuri and Sadighi et al. applied the Adomian decomposition method to Schrödinger equation [18, 25]; Biazar et al., He, Mousaa et al. studied the linear and nonlinear Schrödinger equations by Homotopy perturbation method [4, 12, 22]; Alomari et al., Ghanbari examined the linear and nonlinear Schrödinger equations by Homotopy analysis method [2, 11]; Hosseinzadeh, Wazwaz analyzed the linear and nonlinear Schrödinger equations by Variational iteration method [13, 29]; Iskenderov et.al., Mahmudov, Yagub et al., Yıldırım Aksoy examined the solvability of

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Received: 7 September 2022; Accepted: 20 September 2022; Published: 30 September 2022
Keywords. (Schrödinger equation, finite difference method, stability)
2010 *Mathematics Subject Classification*. 65M06, 65M12
Cited this article as: Yıldırım Aksoy N. and Sarıahmet E. On the Stability of Finite Difference Scheme for the Schrödinger Equation Including Momentum Operator, Turkish Journal of Science.2022, 7(2), 107-115.

Schrödinger equations by Galerkin's method [15, 16, 21, 31–33]. Besides, there is a great variety of solution procedure for Schrödinger equation.

In this work, we apply the finite difference method to a linear Schrödinger equation. In studies [3, 7, 8, 10, 16, 27], the solutions of linear Schrödinger equations is examined by finite difference method and, in that studies, generally, the stability and convergence of difference scheme are shown. Also, in studies [5, 9, 14, 17, 23, 24, 26, 28, 30] the finite difference method is applied to the boundary value problems for nonlinear Schrödinger equations and in most of them, the stability, error and convergence of method are analyzed.

In the most of studies mentioned above, Schrödinger equations do not include the momentum operator. Especially, [27], the numerical solution of linear Schrödinger equation including a momentum operator is investigated. For this, the finite difference method is applied to the considered problem and the conditionally stability of method is proved. As distinct from the earlier studies in literature, in this work, we examine a boundary value problem for the linear Schrödinger equation including a momentum operator and apply the finite difference method to it. We analyze the difference scheme and prove that scheme is unconditionally stable.

Consider the following problem for linear Schrödinger equation including a momentum operator;

$$i\frac{\partial u}{\partial t} + a_0\frac{\partial^2 u}{\partial \varsigma^2} + ia_1\frac{\partial u}{\partial \varsigma} - a_2(\varsigma)u + a_3(\varsigma)u = g(\varsigma,t),\ (\varsigma,t) \in \Omega \tag{1}$$

$$u(\varsigma,0) = f(\varsigma),\ \varsigma \in I \tag{2}$$

$$u(0,t) = u(X,t) = 0,\ t \in (0,T) \tag{3}$$

where $I = (0, X)$, $\Omega = I \times (0, T)$, $a_0, a_1 > 0$ are real numbers; $a_2(\varsigma)$ and $a_3(\varsigma)$ are real valued functions such that

$$0 < a_2(\varsigma) \leq \mu_0 \textit{ almost everywhere (a.e.) in } I,\ \mu_0 = const. > 0\,, \tag{4}$$

$$a_3 \in L_2(I),\ |a_3(\varsigma)| \leq b_0 \textit{ a.e. in } I, \tag{5}$$

$b_0 > 0$ is a given number; $f \in \mathring{W}_2^2(I)$, $g \in W_2^{0,1}(\Omega)$.

Here, $L_\infty(I)$ is the space of all functions that are essentially bounded on I equipped with the norm $\|u\|_{L_\infty(I)} = ess\sup_I |u|$;

$$W_p^r(\Omega) \equiv \left\{ \begin{array}{l} u \in L_p(\Omega) : D^\gamma u \in L_p(\Omega) \text{ for every multi-index } \gamma \text{ with } |\gamma| \leq r, \\ \text{where } D^\gamma u \text{ is the weak(or distributional) partial derivative} \end{array} \right\}$$

and

$$\mathring{W}_p^r(\Omega) \equiv \text{the closure of } C_0^\infty(\Omega) \text{ in the space } W_p^r(\Omega)$$

[1].

In [19], it was shown that the following theorem is valid:

Theorem 1.1. *Assume that (4) and (5) are satisfied and $f \in \mathring{W}_2^2(I)$, $g \in W_2^{0,1}(\Omega)$. Then there exists a unique solution $u \in \mathring{W}_2^{2,1}(\Omega)$ of the problem (1)-(3) and the following estimate holds*

$$\|u(.,t)\|_{\mathring{W}_2^{2,1}(\Omega)} \leq c_0(\|f\|_{\mathring{W}_2^2(I)} + \|g\|_{W_2^{0,1}(\Omega)}) \tag{6}$$

where $c_0 > 0$ is a constant independent of f, g.

2. NOTATIONS AND DIFFERENCE SCHEME

In this section, we will denote the notations used in the paper and discretize the problem (1)-(3). Later, we will express some lemmas and inequalities used in the paper.

Let α, β be any positive integers, $h = \frac{X}{\alpha-1}$, $\tau = \frac{T}{\beta}$,

$$\begin{aligned}
\Omega_h &= \left\{ \varsigma_k : \varsigma_k = kh - \frac{h}{2},\ k = \overline{1, \alpha-1},\ \varsigma_1 - \frac{h}{2} = 0,\ \varsigma_{\alpha-1} + \frac{h}{2} = X, \right\}, \\
\Omega_\tau &= \left\{ t_l : t_l = l\tau,\ l = \overline{0, \beta} \right\}, \\
\Omega_h^\tau &= \Omega_h \times \Omega_\tau .
\end{aligned}$$

Let $u_{kl},\ k = \overline{0, \alpha}, l = \overline{0, \beta}$ be the numerical approximation of $u(\varsigma, t)$ at the point (ς_k, t_l) on Ω_h^τ.
Introduce the following notations:

$$\begin{aligned}
\delta_{\bar{t}} u_{kl} &= \frac{u_{kl} - u_{kl-1}}{\tau},\ \delta_{\bar{\varsigma}} u_{kl} = \frac{u_{kl} - u_{k-1l}}{h}, \\
\delta_{\varsigma} u_{kl} &= \frac{u_{k+1l} - u_{kl}}{h},\ \delta_{\varsigma\bar{\varsigma}} u_{kl} = \frac{\delta_\varsigma u_{kl} - \delta_{\bar{\varsigma}} u_{kl}}{h} = \frac{u_{k+1l} - 2u_{kl} + u_{k-1l}}{h^2}, \\
(v, w) &= h \sum_{k=1}^{\alpha-1} v_k \overline{w}_k,\ \|v\|_p = \sqrt[p]{h \sum_{k=1}^{\alpha-1} |v_k|^p},\ \|v\|_\infty = \max_{1 \le k \le \alpha-1} |v_k|,\ \left\|\delta_\varsigma v\right\|_2 = \sqrt{h \sum_{k=1}^{\alpha-1} \left|\delta_\varsigma v_k\right|^2}
\end{aligned}$$

where $v, w \in V_h = \{v : v = (v_1, v_2, ..., v_{\alpha-1})\}$ are discrete grid functions on Ω_h. We denote by $\|.\|_2$, $\|.\|_\infty$, $(.,.)$ the discrete norms on spaces $L_2(I)$, $L_\infty(I)$ and discrete inner product on $L_2(I)$, respectively. Also, throughout this paper, we denote by $c_k = 1, 2, ..., 5$ the positive constants independent from τ, h and m.

Now, we present finite difference scheme of problem (1)-(3) as follows:

$$i\delta_{\bar{t}} u_{kl} + a_0 \delta_{\varsigma\bar{\varsigma}} u_{kl} + ia_1 \delta_{\bar{\varsigma}} u_{kl} - a_{2k} u_{kl} + a_{3k} u_{kl} = g_{kl},\ k = \overline{1, \alpha-1}, l = \overline{1, \beta}, \tag{7}$$

$$u_{k0} = f_k,\ k = \overline{0, \alpha}, \tag{8}$$

$$u_{0l} = u_{\alpha l} = 0,\ l = \overline{1, \beta}, \tag{9}$$

where the grid functions a_{2k}, a_{3k}, g_{kl} and f_k are Steklov averages of the functions $a_2(\varsigma)$, $a_3(\varsigma)$, $g(\varsigma, t)$ and $f(\varsigma)$ respectively, defined by

$$\begin{aligned}
a_{2k} &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} a_2(\varsigma) d\varsigma,\ k = \overline{1, \alpha-1} \\
a_{3k} &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} a_3(\varsigma) d\varsigma,\ k = \overline{1, \alpha-1} \\
g_{kl} &= \frac{1}{\tau h} \int_{t_{l-1}}^{t_l} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} g(\varsigma, t) d\varsigma dt,\ k = \overline{1, \alpha-1},\ l = \overline{1, \beta} \\
f_k &= \frac{1}{h} \int_{\varsigma_k - h/2}^{\varsigma_k + h/2} f(\varsigma) d\varsigma,\ k = \overline{1, \alpha-1},\ f_0 = f_\alpha = 0
\end{aligned}$$

[6]. Also, from conditions (4) and (5), the inequalities

$$0 \le a_{2k} \le \mu_0,\ k = \overline{1, \alpha-1}, \tag{10}$$

$$0 \le |a_{3k}| \le b_0,\ k = \overline{1, \alpha-1} \tag{11}$$

is written.

In the paper, the lemmas and inequalities we need are as follows:

Lemma 2.1. *(Discrete Gronwall's Inequality [9]): Assume that the nonnegative grid functions* $\{w(z), y(z),\ z = 1, 2, ..., \beta,\ \beta\tau = T\}$ *satisfy the inequality*

$$w(z) \leq y(z) + \tau \sum_{\iota=1}^{z} B_\iota w(\iota),$$

where B_ι $(\iota = 1, 2, ..., \beta)$ *are nonnegative constant. Then, for any* $0 \leq z \leq \beta$*, there is*

$$w(z) \leq y(z) \exp\left(z\tau \sum_{\iota=1}^{z} B_\iota \right).$$

Lemma 2.2. *(Summation by Parts Formula): For any two grid functions*

$v, w \in V_h = \{v : v = (v_0, v_1, v_2, ..., v_{\alpha-1}, v_\alpha),\ v_0 = v_\alpha = 0\}$, we have

$$h \sum_{k=1}^{\alpha-1} \left(\delta_{\varsigma\bar{\varsigma}} v_k\right) \overline{w}_k = -h \sum_{k=1}^{\alpha} \left(\delta_{\bar{\varsigma}} v_k\right) \left(\delta_{\bar{\varsigma}} \overline{w}_k\right).$$

Lemma 2.3. *($\in$ −Cauchy's inequality [20]): For any* $\in > 0$ *and arbitrary a and b, the inequality*

$$ab \leq \frac{\in}{2} a^2 + \frac{1}{2\in} b^2$$

is valid.

Lemma 2.4. *(Young's İnequality): Let* $a, b \geq 0$*. Then,*

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$$

when $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \in (1, +\infty)$.

3. THE STABILITY OF DIFFERENCE SCHEME

In this section, firstly, we obtain an estimate for solution of scheme (7)-(9). Later, using this estimate we prove the stability of scheme.

Theorem 3.1. *Assume that (4) and (5) are satisfied and* $f \in \mathring{W}_2^2(I)$, $g \in W_2^{0,1}(\Omega)$*. Then, the solution* u_{km} *of scheme (7)-(9) for any* $m \in \{1, 2, ..., \beta\}$ *satisfies the estimate*

$$h \sum_{k=1}^{\alpha-1} |u_{km}|^2 + 2h \sum_{l=1}^{m} \sum_{k=1}^{\alpha-1} |u_{kl} - u_{kl-1}|^2 + 2a_1\tau \sum_{l=1}^{m} |u_{\alpha-1l}|^2 + 2a_1\tau \sum_{l=1}^{m} \sum_{k=1}^{\alpha-1} |u_{kl} - u_{k-1l}|^2 \leq$$
$$c_1 \left(h \sum_{k=1}^{\alpha-1} \left|f_k\right|^2 + \tau h \sum_{l=1}^{\beta} \sum_{k=1}^{\alpha-1} \left|g_{kl}\right|^2 \right). \tag{12}$$

Proof. For any grid function ξ_{kl} defined on Ω_h^τ with conditions $\xi_{0l} = \xi_{\alpha l} = 0$ for $l = \overline{1, \beta}$, scheme (7)-(9) is equivalent to the summation identity

$$ih \sum_{k=1}^{\alpha-1} \delta_{\bar{t}} u_{kl} \overline{\xi}_{kl} + a_0 h \sum_{k=1}^{\alpha-1} \delta_{\varsigma\bar{\varsigma}} u_{kl} \overline{\xi}_{kl} + ia_1 h \sum_{k=1}^{\alpha-1} \delta_{\bar{\varsigma}} u_{kl} \overline{\xi}_{kl} -$$
$$h \sum_{k=1}^{\alpha-1} a_{2k} u_{kl} \overline{\xi}_{kl} + h \sum_{k=1}^{\alpha-1} a_{3k} u_{kl} \overline{\xi}_{kl} = h \sum_{k=1}^{\alpha-1} g_{kl} \overline{\xi}_{kl}, \tag{13}$$

where $\overline{\xi}_{kl}$ is the conjugate of ξ_{kl}. If we substitute $\tau\overline{u}_{kl}$ for $\overline{\xi}_{kl}$ in (13) and apply the formula of summation by parts, we get

$$\begin{aligned} & ih\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{t}}u_{kl}\overline{u}_{kl} - a_0h\tau\sum_{k=1}^{\alpha-1}\left|\delta_{\bar{\varsigma}}u_{kl}\right|^2 + ia_1h\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{\varsigma}}u_{kl}\overline{u}_{kl} - \\ & h\tau\sum_{k=1}^{\alpha-1}a_{2k}\left|u_{kl}\right|^2 + h\tau\sum_{k=1}^{\alpha-1}a_{3k}\left|u_{kl}\right|^2 = h\tau\sum_{k=1}^{\alpha-1}g_{kl}\overline{u}_{kl}. \end{aligned} \tag{14}$$

If we extract its complex conjugate from (14) and then, use the relations

$$\tau\left(\delta_{\bar{t}}u_{kl}\overline{u}_{kl} + \delta_{\bar{t}}\overline{u}_{kl}u_{kl}\right) = |u_{kl}|^2 - |u_{kl-1}|^2 + |u_{kl} - u_{kl-1}|^2 \tag{15}$$

$$h\left(\delta_{\bar{\varsigma}}u_{kl}\overline{u}_{kl} + \delta_{\bar{\varsigma}}\overline{u}_{kl}u_{kl}\right) = |u_{kl}|^2 - |u_{k-1l}|^2 + |u_{kl} - u_{k-1l}|^2 \tag{16}$$

we get

$$\begin{aligned} & h\sum_{k=1}^{\alpha-1}\left(|u_{kl}|^2 - |u_{kl-1}|^2 + |u_{kl} - u_{kl-1}|^2\right) + a_1\tau\sum_{k=1}^{\alpha-1}\left(|u_{kl}|^2 - |u_{k-1l}|^2 + |u_{kl} - u_{k-1l}|^2\right) = \\ & 2h\tau\sum_{k=1}^{\alpha-1}Im\left(g_{kl}\overline{u}_{kl}\right)\ for\ l = \overline{1,\beta}. \end{aligned} \tag{17}$$

If we sum all equalities in (17) in l from 1 to $m \le \beta$ and consider

$$\begin{aligned} \sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left(|u_{kl}|^2 - |u_{kl-1}|^2\right) &= \sum_{k=1}^{\alpha-1}\left(|u_{km}|^2 - |u_{k0}|^2\right) = \sum_{k=1}^{\alpha-1}|u_{km}|^2 - \sum_{k=1}^{\alpha-1}\left|f_k\right|^2 \\ \sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left(|u_{kl}|^2 - |u_{k-1l}|^2\right) &= \sum_{l=1}^{m}\left(|u_{\alpha-1l}|^2 - |u_{0l}|^2\right) = \sum_{l=1}^{m}|u_{\alpha-1l}|^2 \end{aligned}$$

by (8) and (9), we obtain from (17) the inequality

$$\begin{aligned} & h\sum_{k=1}^{\alpha-1}|u_{km}|^2 + h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl} - u_{kl-1}|^2 + a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2 + \\ & a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl} - u_{k-1l}|^2 \le 2h\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right||u_{kl}| + h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2. \end{aligned}$$

Let's distinguish m-th term from first summation in the right-hand side (r.h.s.) of above inequality and apply $\epsilon - Cauchy's$ inequality to distinguished term. Then, if we take $\epsilon = 2\tau$ and use Young's inequality we get

$$\begin{aligned} & h\sum_{k=1}^{\alpha-1}|u_{km}|^2 + 2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl} - u_{kl-1}|^2 + 2a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2 + 2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl} - u_{k-1l}|^2 \le \\ & 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2 + 4T\tau h\sum_{k=1}^{\alpha-1}\left|g_{km}\right|^2 + 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|u_{kl}|^2 + 2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2 \end{aligned}$$

which is equal to

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2+2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{kl-1}|^2+2a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2+2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{k-1l}|^2\leq$$
$$4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2+2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|u_{kl}|^2+2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2 \tag{18}$$

for *any* $m\in\{1,2,...,\beta\}$. Since all terms in the l.h.s. of (18) are non-negative, it is written that

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2\leq 4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2+2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|u_{kl}|^2+2h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2. \tag{19}$$

In (19), using discrete Gronwall's Inequality, we obtain

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2\leq c_2\left(h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2+\tau h\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2\right)\text{ for } any\ m\in\{1,2,...,\beta\}. \tag{20}$$

If we use the inequality (20) in (18), we get for *any* $m\in\{1,2,...,\beta\}$

$$h\sum_{k=1}^{\alpha-1}|u_{km}|^2+2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{kl-1}|^2+2a_1\tau\sum_{l=1}^{m}|u_{\alpha-1l}|^2+2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|u_{kl}-u_{k-1l}|^2\leq$$
$$c_3\left(h\sum_{k=1}^{\alpha-1}\left|f_k\right|^2+\tau h\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}\right|^2\right) \tag{21}$$

which shows the hypothesis of theorem 3.1 is valid. □

Theorem 3.2. *Suppose that u^1_{kl}, is a solution corresponding to the initial value f^1_k and the right side g^1_{kl} of scheme (7)-(9) and u^2_{kl} is a solution corresponding to the initial value f^2_k and the right side g^2_{kl} of scheme (7)-(9). Assume that the conditions of theorem 3.1 are fulfilled. Let $\Phi_{kl}=u^1_{kl}-u^2_{kl}$. Then, for any $m\in\{1,2,...,\beta\}$ and $h,\tau>0$*

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2\leq c_4\left(h\sum_{k=1}^{\alpha-1}\left|f^1_k-f^2_k\right|^2+h\tau\sum_{l=1}^{\beta-1}\sum_{k=1}^{\alpha-1}\left|g^1_{kl}-g^2_{kl}\right|^2\right).$$

Hence, the difference scheme (7)-(9) is unconditionally stable.

Proof. It is clear that Φ_{kl} satisfies the scheme

$$i\delta_{\bar{t}}\Phi_{kl}+a_0\delta_{\varsigma\bar{\varsigma}}\Phi_{kl}+ia_1\delta_{\bar{\varsigma}}\Phi_{kl}-a_{2k}\Phi_{kl}+a_{3k}\Phi_{kl}=g^1_{kl}-g^2_{kl},\ \ k=\overline{1,\alpha-1}, l=\overline{1,\beta}$$
$$\Phi_{k0}=f^1_k-f^2_k,\ \ k=\overline{0,\alpha}$$
$$\Phi_{0l}=\Phi_{\alpha l}=0,\ \ l=\overline{1,\beta}$$

which is equivalent to

$$ih\sum_{k=1}^{\alpha-1}\delta_{\bar{t}}\Phi_{kl}\overline{\Theta}_{kl}+a_0h\sum_{k=1}^{\alpha-1}\delta_{\varsigma\bar{\varsigma}}\Phi_{kl}\overline{\Theta}_{kl}+ia_1h\sum_{k=1}^{\alpha-1}\delta_{\bar{\varsigma}}\Phi_{kl}\overline{\Theta}_{kl}-$$
$$h\sum_{k=1}^{\alpha-1}a_{2k}\Phi_{kl}\overline{\Theta}_{kl}+h\sum_{k=1}^{\alpha-1}a_{3k}\Phi_{kl}\overline{\Theta}_{kl}=h\sum_{k=1}^{\alpha-1}\left(g^1_{kl}-g^2_{kl}\right)\overline{\Theta}_{kl}, \tag{22}$$

for any grid function $\overline{\Theta}_{kl}$, where $\overline{\Theta}_{kl}$ is the conjugate of Θ_{kl} defined on Ω_h^τ such that $\Theta_{0l} = \Theta_{\alpha l} = 0$ for $l = \overline{1,\beta}$. From (22) for $\overline{\Theta}_{kl} = \tau\overline{\Phi}_{kl}$ it is written that

$$ih\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{t}}\Phi_{kl}\overline{\Phi}_{kl} - a_0 h\tau\sum_{k=1}^{\alpha-1}\left|\delta_{\bar{\varsigma}}\Phi_{kl}\right|^2 + ia_1 h\tau\sum_{k=1}^{\alpha-1}\delta_{\bar{\varsigma}}\Phi_{kl}\overline{\Phi}_{kl} -$$
$$h\tau\sum_{k=1}^{\alpha-1}a_{2k}\left|\Phi_{kl}\right|^2 + h\tau\sum_{k=1}^{\alpha-1}a_{3k}\left|\Phi_{kl}\right|^2 = h\tau\sum_{k=1}^{\alpha-1}\left(g_{kl}^1 - g_{kl}^2\right)\overline{\Phi}_{kl}. \tag{23}$$

with summation by parts. Extracting its complex conjugate from (23) and using (15) and (16) for Φ_{kl}, we obtain

$$h\sum_{k=1}^{\alpha-1}\left(|\Phi_{kl}|^2 - |\Phi_{kl-1}|^2 + |\Phi_{kl} - \Phi_{kl-1}|^2\right) + a_1\tau\sum_{k=1}^{\alpha-1}\left(|\Phi_{kl}|^2 - |\Phi_{k-1l}|^2 + |\Phi_{kl} - \Phi_{k-1l}|^2\right) =$$
$$2h\tau\sum_{k=1}^{\alpha-1}Im\left(\left(g_{kl}^1 - g_{kl}^2\right)\overline{\Phi}_{kl}\right)\ for\ l = \overline{1,\beta}. \tag{24}$$

Summing all equalities in (24) in l from 1 to $m \le \beta$ and using $\Phi_{k0} = f_k^1 - f_k^2$ for $k = \overline{0,\alpha}$, $\Phi_{0l} = 0$ for $l = \overline{1,\beta}$, we have

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 +$$
$$a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le 2h\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right||\Phi_{kl}| + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2.$$

which is equal to

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 +$$
$$a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le 2h\tau\sum_{k=1}^{\alpha-1}\left|g_{km}^1 - g_{km}^2\right||\Phi_{km}| +$$
$$2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right||\Phi_{kl}| + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2. \tag{25}$$

Applying $\epsilon - Cauchy's$ and Young's inequalities to (25), we get

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 + 2h\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{kl-1}|^2 + 2a_1\tau\sum_{l=1}^{m}|\Phi_{\alpha-1l}|^2 + 2a_1\tau\sum_{l=1}^{m}\sum_{k=1}^{\alpha-1}|\Phi_{kl} - \Phi_{k-1l}|^2 \le$$
$$4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right|^2 + 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|\Phi_{kl}|^2 + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2 \tag{26}$$

by $\epsilon = 2\tau$. It is clear that all terms in the l.h.s. of (26) are non-negative. So, we write that

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 \le 4Th\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1 - g_{kl}^2\right|^2 + 2h\tau\sum_{l=1}^{m-1}\sum_{k=1}^{\alpha-1}|\Phi_{kl}|^2 + h\sum_{k=1}^{\alpha-1}\left|f_k^1 - f_k^2\right|^2. \tag{27}$$

Thus, applying discrete Gronwall's inequality to (27), we obtain

$$h\sum_{k=1}^{\alpha-1}|\Phi_{km}|^2 \le c_5\left(h\tau\sum_{l=1}^{\beta}\sum_{k=1}^{\alpha-1}\left|g_{kl}^1-g_{kl}^2\right|^2+h\sum_{k=1}^{\alpha-1}\left|f_k^1-f_k^2\right|^2\right) \text{ for } any\ m\in\{1,2,...,\beta\}$$

which this complete the proof. □

4. Conclusion

In this paper, a finite difference scheme for the Schrödinger type equation has been introduced and analyzed. We have obtained a priori estimate for solution of scheme. We have also proved that the proposed scheme is unconditionally stable, without any restriction on both time and spatial step sizes.

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