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Some Characterizations of PS-Statistical Manifolds

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Abstract. In the present study, firstly we state symmetry properties for curvatures of a statistical manifold and give some relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S. After, by defining the notion of para-Sasakian statistical manifold, we give the necessary and sufficient conditions for a structure (D, h, Ψ, w, ζ) to be a para-Sasakian structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. Also, we give some results for curvatures R, R^*, R^S and Ricci tensor of these curvatures on a para-Sasakian statistical manifold. We construct an example of para-Sasakian statistical manifold of dimension 3. Finally, we examined the Einsteinian of para-Sasakian statistical manifolds according to certain conditions.

1. INTRODUCTION

The theory of statistical manifolds, (at the same time it is called information geometry), has started with a study in 1945, where a statistical model was considered as a Riemannian manifold with the tensor given by the Fisher information matrix [15]. After that, the information geometry, which is typically deals with the study of various geometric structures on a statistical manifold, has begun as a study of the geometric structures possessed by a statistical model of probability distributions.

The notion of dual connection, which is also called conjugate connection in affine geometry, has been first introduced into statistics by Amari in 1985 [2]. A statistical model equipped with a Riemannian metric together with a pair of dual affine connections is called a *statistical manifold*. For details about statistical manifolds and information geometry, one can see [3], [5], [6], [10], [11], [12], [13], [14], [19] and etc.

Also, if Ψ is a tensor field of type $(1, 1)$, w is a 1-form and ζ is a vector field on a $(2n + 1)$-dimensional differentiable manifold M, then almost contact structure (Ψ, w, ζ) which is related to almost complex structures and satisfies the conditions $\Psi^2 = -I + w \otimes \zeta$, $w(\zeta) = 1$ has been determined by Sasaki in 1960 [16]. After in 1976, on an n-dimensional differentiable manifold M, almost paracontact structure which is a similar structure with almost contact structure, related to almost product structures and satisfies the conditions $\Psi^2 = I - w \otimes \zeta$, $w(\zeta) = 1$ has been determined by Sato [17]. With the aid of these definitions, different types of manifolds have been defined and studied by many mathematicians.

According to these notions, nowadays lots of studies have been started to be done by scientists. For example, in [22] the authors have defined the concept of quaternionic Kähler-like statistical manifold and derived the main properties of quaternionic Kähler-like statistical submersions, extending in a new setting some previous results obtained by K. Takano concerning statistical manifolds endowed with almost complex

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(in [20]) and almost contact structures (in [21]). In [7], the authors have introduced the notion of Sasakian statistical structure and obtained the condition for a real hypersurface in a holomorphic statistical manifold to admit such a structure. Also in [8], the notion of a Kenmotsu statistical manifold is introduced, which is locally obtained as the warped product of a holomorphic statistical manifold and a line by authors. And they have showed that, a Kenmotsu statistical manifold of constant Ψ-sectional curvature is constructed from a special Kahler manifold, which is an important example of holomorphic statistical manifold.

In this paper, after giving some basic notions about statistical structures and para-Sasakian manifolds in Preliminaries, in Section 3 we give symmetry properties of curvatures R and R^* which are the curvatures of the connections D and D^*, respectively and R^S which is statistical curvature of a statistical manifold and obtain some results for relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S. In fourth section, we define the notion of para-Sasakian statistical manifold and give the necessary and sufficient coditions for a structure (D, h, Ψ, w, ζ) to be a para-Sasakian structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. Also, we give some results about the curvatures R, R^*, R^S and Ricci tensor of these curvatures on a para-Sasakian statistical manifold. We construct an example of 3-dimensional para-Sasakian statistical manifold and give its all of connections and components of curvature tensors. And in the fifth section, we study on Ricci semi-symmetric and Ricci pseudo-symmetric para-Sasakian statistical manifolds and after we give some characterizations for ζ-projectively flat, projectively flat and Ψ-projectively semi-symmetric para-Sasakian statistical manifolds.

2. PRELIMINARIES

In this section, we recall some notions about statistical structures and para-Sasakian manifolds, respectively. Througout this paper, we suppose that M is an n-dimensional manifold, h is a Riemannian metric and $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p, q) on M.

On M, a parametric family of torsion-free connections $D^{(\alpha)}$ indexed by $\alpha \in \mathbb{R}$ can be defined by

$$D^{(\alpha)} = \frac{1+\alpha}{2}D - \frac{1-\alpha}{2}D^*, \tag{1}$$

with

$$D^{(1)} = D,\ D^{(-1)} = D^*,\ D^{(0)} = \frac{1}{2}(D + D^*) := \hat{D}. \tag{2}$$

Here $\hat{D}$ denotes the Levi-Civita (L-C) connection associated with h.

Also, a pair (D, h) is called a *statistical structure* on M, if D is torsion-free and

$$(D_{\Omega_1}h)(\Omega_2, \Omega_3) = (D_{\Omega_2}h)(\Omega_1, \Omega_3),\ \forall\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)}) \tag{3}$$

holds, where the equation (3) is generally called *Codazzi equation*. In this case (M, D, h) is called a *statistical manifold.*

If (D, h) is a statistical structure on M, then the connection D^* which is given by

$$\Omega_1 h(\Omega_2, \Omega_3) = h(D_{\Omega_1}\Omega_2, \Omega_3) + h(\Omega_2, D^*_{\Omega_1}\Omega_3) \tag{4}$$

is called *conjugate or dual connection* of D with respect to h. If (D, h) is a statistical structure on M, then (D^*, h) is a statistical structure on M, too.

For a statistical structure (D, h), the *difference tensor field* $\kappa \in \Gamma(TM^{(1,2)})$ can be defined as

$$\kappa(\Omega_1, \Omega_2) = D_{\Omega_1}\Omega_2 - \hat{D}_{\Omega_1}\Omega_2,\ \forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)}). \tag{5}$$

Moreover, κ satisfies

$$\kappa(\Omega_1, \Omega_2) = \kappa(\Omega_2, \Omega_1), \tag{6}$$

$$\tilde{\kappa}(\Omega_1, \Omega_2, \Omega_3) = h(\kappa(\Omega_1, \Omega_2), \Omega_3) = h(\Omega_2, \kappa(\Omega_1, \Omega_3)) = \tilde{\kappa}(\Omega_1, \Omega_3, \Omega_2), \tag{7}$$

where $\tilde{\kappa} \in \Gamma(TM^{(0,3)})$. Furthermore, we have

$$\kappa = \hat{D} - D^* = \frac{1}{2}(D - D^*). \tag{8}$$

For a more detailed treatment, we refer to [6], [7] and [23].

Now, let us recall some fundamental informations about para-Sasakian manifolds.

A differentiable manifold M is said to admit an *almost paracontact Riemannian structure* (Ψ, w, ζ, h), where Ψ is a tensor field of type (1,1), ζ is a vector field, w is a 1-form and h is a Riemannian metric on M such that

$$\begin{aligned} &\Psi\zeta = 0,\ w(\zeta) = 1,\ h(\zeta, \Omega_1) = w(\Omega_1), \\ &\Psi^2\Omega_1 = \Omega_1 - w(\Omega_1)\zeta, \\ &h(\Psi\Omega_1, \Psi\Omega_2) = h(\Omega_1, \Omega_2) - w(\Omega_1)w(\Omega_2), \end{aligned} \tag{9}$$

for any vector fields Ω_1, Ω_2 on M. In addition, if (Ψ, w, ζ, h) satisfy the equations

$$d\eta = 0,\ \hat{D}_{\Omega_1}\zeta = \Psi\Omega_1, \tag{10}$$

$$(\hat{D}_{\Omega_1}\Psi)\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2w(\Omega_1)w(\Omega_2)\zeta, \tag{11}$$

then M is called a *para-Sasakian (PS) manifold*. On a PS-manifold, for $\forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$ we have the following equations:

$$\widehat{Ric}(\Omega_1, \zeta) = (1 - n)w(\Omega_1), \tag{12}$$

$$\widehat{Q}\zeta = (1 - n)\zeta, \tag{13}$$

$$\widehat{R}(\Omega_1, \Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1, \tag{14}$$

$$\widehat{R}(\zeta, \Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1, \Omega_2)\zeta, \tag{15}$$

$$\widehat{R}(\zeta, \Omega_1)\zeta = \Omega_1 - w(\Omega_1)\zeta, \tag{16}$$

$$w(\widehat{R}(\Omega_1, \Omega_2)\Omega_3) = h(\Omega_1, \Omega_3)w(\Omega_2) - h(\Omega_2, \Omega_3)w(\Omega_1), \tag{17}$$

$$\widehat{Ric}(\Psi\Omega_1, \Psi\Omega_2) = \widehat{Ric}(\Omega_1, \Omega_2) - (1 - n)w(\Omega_1)w(\Omega_2), \tag{18}$$

where $\widehat{R}$, $\widehat{Ric}$ and $\widehat{Q}$ denotes the Riemannian curvature tensor, Ricci tensor and Ricci operator of L-C connection $\hat{D}$, respectively (for detail, see [1], [17] and [18]).

3. R, R^* AND R^S CURVATURES OF STATISTICAL MANIFOLDS

In this section, firstly we recall symmetry properties of curvatures R, R^* and give these properties for R^S. After, we give some results for relations between the Riemannian curvature $\widehat{R}$ and the curvatures R, R^* and R^S.

Lemma 3.1. *Let (M, D, h) be a statistical manifold. Then, the curvatures R and R^* satisfy the following symmetry properties:*

i) $R(\Omega_1, \Omega_2)\Omega_3 + R(\Omega_2, \Omega_3)\Omega_1 + R(\Omega_3, \Omega_1)\Omega_2 = 0,$
$R^*(\Omega_1, \Omega_2)\Omega_3 + R^*(\Omega_2, \Omega_3)\Omega_1 + R^*(\Omega_3, \Omega_1)\Omega_2 = 0;$

ii) $\mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 2h((\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4), \Omega_3),$
$\mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 2h((\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4), \Omega_3);$

iii) $\begin{matrix} \mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0, \\ \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}^*(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0 \end{matrix}$ *if* $(\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_4) = (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_4),$

where $\mathcal{R}$ and $\mathcal{R}^* \in \Gamma(TM^{(0,4)})$ are Riemannian-Christoffel curvature tensors of R and R^*, respectively and they are defined by $h(R(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ and $h(R^*(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}^*(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, for $\forall\Omega_1, \Omega_2, \Omega_3, \Omega_4 \in \Gamma(TM^{(1,0)})$.

Proof. The proof can be found in [9]. □

In [7], the authors have defined a curvature tensor field $S \in \Gamma(TM^{(1,3)})$ as

$$S(\Omega_1, \Omega_2)\Omega_3 = \frac{1}{2}\{R(\Omega_1, \Omega_2)\Omega_3 + R^*(\Omega_1, \Omega_2)\Omega_3\}, \tag{19}$$

for $\forall \Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$ and they have called it *statistical curvature tensor field* of (D, h). Hereafter, in our results we'll denote the statistical curvature tensor field S by R^S. So, let us give the following Theorem which gives the symmetry properties of R^S:

Theorem 3.2. *Let (M, D, h) be a statistical manifold. Then, the statistical curvature tensor field R^S satisfies the following symmetry properties:*
i) $R^S(\Omega_1, \Omega_2)\Omega_3 + R^S(\Omega_2, \Omega_3)\Omega_1 + R^S(\Omega_3, \Omega_1)\Omega_2 = 0,$
ii) $\mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4) + \mathcal{R}^S(\Omega_1, \Omega_2, \Omega_4, \Omega_3) = 0,$
iii) $\mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4) - \mathcal{R}^S(\Omega_3, \Omega_4, \Omega_1, \Omega_2) = 0,$
where $\mathcal{R}^S \in \Gamma(TM^{(0,4)})$ is Riemannian-Christoffel curvature tensor of R^S and it is defined by
$h(R^S(\Omega_1, \Omega_2)\Omega_3, \Omega_4) = \mathcal{R}^S(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, for $\forall \Omega_1, \Omega_2, \Omega_3, \Omega_4 \in \Gamma(TM^{(1,0)})$.

Proof. Using Lemma 3.1-(i) in (19), we get (i). Using Lemma 3.1-(ii) in (19), we reach (ii). And finally, from (i) and (ii), we have (iii). □

Also, we can give the following relations, which have been stated in [9] too, between Riemannian curvature $\widehat{R}$ and the curvatures R, R^* when (M, D, h) is a statistical manifold and we give these relations for R^S.

Using $D = \hat{D} + \kappa$ in $R(\Omega_1, \Omega_2)\Omega_3 = D_{\Omega_1} D_{\Omega_2}\Omega_3 - D_{\Omega_2} D_{\Omega_1}\Omega_3 - D_{[\Omega_1, \Omega_2]}\Omega_3$, we have

$$R(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + (D_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) + \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{20}$$

Again using $D = \hat{D} + \kappa$ in (20), we get

$$R(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{21}$$

Thus, from (20) and (21) we can write

$$(D_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) = 2\{\kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))\}. \tag{22}$$

Similarly, using $D^* = \hat{D} - \kappa$ in $R^*(\Omega_1, \Omega_2)\Omega_3 = D^*_{\Omega_1} D^*_{\Omega_2}\Omega_3 - D^*_{\Omega_2} D^*_{\Omega_1}\Omega_3 - D^*_{[\Omega_1, \Omega_2]}\Omega_3$, we have

$$R^*(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 - (D^*_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (D^*_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) + \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)) \tag{23}$$

and again using $D^* = \hat{D} - \kappa$ in (23), we get

$$R^*(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{24}$$

So, from (23) and (24) we can write

$$(D^*_{\Omega_1}\kappa)(\Omega_2, \Omega_3) - (D^*_{\Omega_2}\kappa)(\Omega_1, \Omega_3) - (\hat{D}_{\Omega_1}\kappa)(\Omega_2, \Omega_3) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1, \Omega_3) = -2\{\kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3))\}. \tag{25}$$

And finally, using (21) and (24) in (19), we reach that

$$R^S(\Omega_1, \Omega_2)\Omega_3 = \widehat{R}(\Omega_1, \Omega_2)\Omega_3 + \kappa(\Omega_1, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\Omega_1, \Omega_3)). \tag{26}$$

4. PARA-SASAKIAN (PS) STATISTICAL MANIFOLDS

In this section, firstly we define the notion of para-Sasakian (PS) statistical manifold and give the necessary and sufficient coditions for a structure (D, h, Ψ, w, ζ) to be a PS-structure when (D, h) is a statistical structure and (Ψ, w, ζ, h) is an almost paracontact Riemannian manifold. After that, we give some results about the curvatures R, R^* and R^S and Ricci tensor of these curvatures on a PS-statistical manifold.

Let Ω be the fundamental 2-form of a PS-manifold (M, Ψ, w, ζ, h) defined by

$$\Omega(\Omega_1, \Omega_2) = h(\Omega_1, \Psi\Omega_2), \tag{27}$$

for $\forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, we have

Lemma 4.1. *Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be an almost paracontact Riemannian structure on M. Then, we have*
i) $(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) = h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3 - \Psi D_{\Omega_1}\Omega_3)$,
ii) $(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) - (D^*_{\Omega_1}\Omega)(\Omega_2, \Omega_3) = -2g(\Omega_2, \kappa(\Omega_1, \Psi\Omega_3) + \Psi\kappa(\Omega_1, \Omega_3))$,
iii) $D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = (\hat{D}_{\Omega_1}\Psi)\Omega_2 + \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2)$,
for $\forall\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$.

Proof. **i)** From (4) and (27), we get

$$\begin{aligned}(D_{\Omega_1}\Omega)(\Omega_2, \Omega_3) &= D_{\Omega_1}\Omega(\Omega_2, \Omega_3) - \Omega(D_{\Omega_1}\Omega_2, \Omega_3) - \Omega(\Omega_2, D_{\Omega_1}\Omega_3)\\ &= \Omega_1 h(\Omega_2, \Psi\Omega_3) - h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) - h(\Omega_2, \Psi D_{\Omega_1}\Omega_3)\\ &= h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) + h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3) - h(D_{\Omega_1}\Omega_2, \Psi\Omega_3) - h(\Omega_2, \Psi D_{\Omega_1}\Omega_3)\\ &= h(\Omega_2, D^*_{\Omega_1}\Psi\Omega_3 - \Psi D_{\Omega_1}\Omega_3).\end{aligned} \tag{28}$$

ii) Substracting the dual of equation (28) from (28) and using (8), the proof completes.
iii) From (5) and (8), we have (iii). □

Definition 4.2. *(D, h, Ψ, w, ζ) is a PS-statistical structure on M, if*
i) *(D, h) is a statistical structure,*
ii) *(Ψ, w, ζ, h) is a PS-structure,*
iii) *for $\forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$, the equation*

$$\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0 \tag{29}$$

is satisfied.

Thus, we can prove the following Theorem:

Theorem 4.3. *Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be an almost paracontact Riemannian structure on M. Then, (D, h, Ψ, w, ζ) is a PS-statistical structure on M iff the equations*

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta \tag{30}$$

and

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + w(D_{\Omega_1}\zeta)\zeta \tag{31}$$

hold for $\forall\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$.

Proof. Let (D, h) be a statistical structure and (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Then, from (11) and Lemma 4.1-(iii), we get

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta + \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2).$$

So, if (D, h, Ψ, w, ζ) is a PS-structure on M, then from (29) we have (30). Also, putting $\Omega_2 = \zeta$ in the expression of the dual of (30), from (9) we obtain (31).

Conversely, let us assume that the equations (30) and (31) hold for $\forall \Omega_1, \Omega_2 \in \Gamma(TM)$. Taking $\Psi\Omega_2$ instead of Ω_2 in the equation (30) and applying Ψ to the resulting equation, from (9) and (31) we have

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta$$

and this is the dual of (30). Finally we have to see that, (Ψ, w, ζ, h) is a PS-structure and the equation (29) holds. Using (5) and (8) in the equation which is in the Lemma 4.1-(iii), we have

$$D_{\Omega_1}\Psi\Omega_2 - \Psi D^*_{\Omega_1}\Omega_2 = \frac{1}{2}\{(D_{\Omega_1}\Psi)\Omega_2 + (D^*_{\Omega_1}\Psi)\Omega_2\} + \frac{1}{2}\{D_{\Omega_1}\Psi\Omega_2 - D^*_{\Omega_1}\Psi\Omega_2\} + \frac{1}{2}\Psi\{D_{\Omega_1}\Omega_2 - D^*_{\Omega_1}\Omega_2\}.$$

Taking the dual of the last equation, we get

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = (\hat{D}_{\Omega_1}\Psi)\Omega_2 - \kappa(\Omega_1, \Psi\Omega_2) - \Psi\kappa(\Omega_1, \Omega_2). \tag{32}$$

The dual of the equation (30), i.e. $(30)^*$, is

$$D^*_{\Omega_1}\Psi\Omega_2 - \Psi D_{\Omega_1}\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta.$$

Thus, from (32) and $(30)^*$ we obtain that

$$-(\hat{D}_{\Omega_1}\Psi)\Omega_2 - h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta = -\kappa(\Omega_1, \Psi\Omega_2) - \Psi\kappa(\Omega_1, \Omega_2). \tag{33}$$

Also, from Lemma 4.1-(iii) and (30), we get

$$-(\hat{D}_{\Omega_1}\Psi)\Omega_2 - h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta = \kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2). \tag{34}$$

So, from (33) and (34) we can reach that, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0$ holds and also we have $(\hat{D}_{\Omega_1}\Psi)\Omega_2 = -h(\Omega_1, \Omega_2)\zeta - w(\Omega_2)\Omega_1 + 2\eta(\Omega_1)w(\Omega_2)\zeta$. Thus, (Ψ, w, ζ, h) is a PS-structure and this completes the proof. □

Example 4.4. *Let (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Set the connection $\check{D}$ as*

$$\check{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)\Omega_2 + w(\Omega_2)\Omega_1 + h(\Omega_1, \Omega_2)\zeta, \tag{35}$$

for any $\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, $\check{D}$ is torsion-free and satisfies the Codazzi equation (3). So, $(\check{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h).

Also, from (5) and (35) we have $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)\Omega_2 + w(\Omega_2)\Omega_1 + h(\Omega_1, \Omega_2)\zeta$. So, for this structure we have

$$\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 2\eta(\Omega_1)\Psi\Omega_2 + w(\Omega_2)\Psi\Omega_1 + h(\Omega_1, \Psi\Omega_2)\zeta.$$

Now, let us suppose that $\kappa(\Omega_1, \Psi\Omega_2) + \Phi\kappa(\Omega_1, \Omega_2) = 0$ is satisfied for this structure. Then, we have

$$2\eta(\Omega_1)\Psi\Omega_2 + w(\Omega_2)\Psi\Omega_1 + h(\Omega_1, \Psi\Omega_2)\zeta = 0.$$

Applying w to the last equation, we get $h(\Omega_1, \Psi\Omega_2) = 0 \Rightarrow \Psi = 0$ and this is a contradiction. So, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2)$ cannot be zero. Hence, $(\check{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h) but it isn't a p-S statistical structure.

Example 4.5. *Let (Ψ, w, ζ, h) be a PS-Riemannian structure on M. Set the connection $\tilde{D}$ as*

$$\tilde{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)w(\Omega_2)\zeta, \tag{36}$$

for any $\Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Then, $\tilde{D}$ is torsion-free and satisfies the Codazzi equation (3). So, $(\tilde{D}, h)$ is a statistical structure on the PS-Riemannian manifold (M, Ψ, w, ζ, h).

Also, from (5) and (36) we have $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)w(\Omega_2)\zeta$. So, $\kappa(\Omega_1, \Psi\Omega_2) + \Psi\kappa(\Omega_1, \Omega_2) = 0$ is satisfied for the connection $\tilde{D}$. Hence $(\tilde{D}, h, \Psi, w, \zeta)$ is a PS-statistical structure on M.

Here, we obtain some results about the curvatures R, R^* and R^S. For this, we'll give some results for a PS-statistical manifold (M, Ψ, w, ζ, h).

Taking $\Omega_2 = \zeta$ in (30) and (30)* and using (9), we have

$$D^*_{\Omega_1}\zeta = \Psi\Omega_1 + w(D^*_{\Omega_1}\zeta)\zeta \tag{37}$$

and

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + w(D_{\Omega_1}\zeta)\zeta, \tag{38}$$

respectively. Also, from (5), (8), (37) and (38) we have

$$\kappa(\Omega_1,\zeta) = D_{\Omega_1}\zeta - \hat{D}_{\Omega_1}\zeta = w(D_{\Omega_1}\zeta)\zeta \tag{39}$$

and

$$\kappa(\Omega_1,\zeta) = \hat{D}_{\Omega_1}\zeta - D^*_{\Omega_1}\zeta = -w(D^*_{\Omega_1}\zeta)\zeta, \tag{40}$$

respectively. Thus, from (37)-(40) we get

$$D^*_{\Omega_1}\zeta = \Psi\Omega_1 - \kappa(\Omega_1,\zeta) \tag{41}$$

$$D_{\Omega_1}\zeta = \Psi\Omega_1 + \kappa(\Omega_1,\zeta) \tag{42}$$

and

$$\Psi\kappa(\Omega_1,\zeta) = 0. \tag{43}$$

Furthermore, from (5), (10), (39) and (42) we have

$$\kappa(\Omega_1,\kappa(\Omega_2,\zeta)) = w(D_{\Omega_1}\zeta)w(D_{\Omega_2}\zeta)\zeta \tag{44}$$

and so, we get

$$\kappa(\Omega_1,\kappa(\Omega_2,\zeta)) = \kappa(\Omega_2,\kappa(\Omega_1,\zeta)). \tag{45}$$

Now, we can give some results about the curvatures R, R^* and R^S.

Using (14), (15) and (45) in (21), we have

$$R(\Omega_1,\Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 + (\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta) \tag{46}$$

and

$$R(\zeta,\Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1,\Omega_2)\zeta + (\hat{D}_{\zeta}\kappa)(\Omega_1,\Omega_2) - (\hat{D}_{\Omega_1}\kappa)(\zeta,\Omega_2) + \kappa(\zeta,\kappa(\Omega_1,\Omega_2)) - \kappa(\Omega_1,\kappa(\zeta,\Omega_2)). \tag{47}$$

From Lemma 3.1-(iv) and (46), we get

$$w(R(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) + h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3). \tag{48}$$

Also, from (21), we have

$$\begin{aligned} w(R(\Omega_1,\Omega_2)\Omega_3) = &-w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) \\ &+ w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3) + \kappa(\Omega_1,\kappa(\Omega_2,\Omega_3)) - \kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \end{aligned} \tag{49}$$

Thus from (48) and (49), we obtain that

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta)-(\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3)-(\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)+\kappa(\Omega_1,\kappa(\Omega_2,\Omega_3))-\kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \tag{50}$$

Similarly, using (14), (15) and (45) in (24), we have

$$R^*(\Omega_1,\Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 - (\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) + (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta) \tag{51}$$

and

$$R^*(\zeta,\Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1,\Omega_2)\zeta - (\hat{D}_\zeta\kappa)(\Omega_1,\Omega_2) + (\hat{D}_{\Omega_1}\kappa)(\zeta,\Omega_2) + \kappa(\zeta,\kappa(\Omega_1,\Omega_2)) - \kappa(\Omega_1,\kappa(\zeta,\Omega_2)). \quad (52)$$

From Lemma 3.1-(iv) and (51), we get

$$w(R^*(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) - h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3). \quad (53)$$

Also, from (24), we have

$$\begin{aligned} w(R^*(\Omega_1,\Omega_2)\Omega_3) &= -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) \\ &\quad - w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3) - \kappa(\Omega_1,\kappa(\Omega_2,\Omega_3)) + \kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \end{aligned} \quad (54)$$

Thus from (53) and (54), we obtain that

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3) - \kappa(\Omega_1,\kappa(\Omega_2,\Omega_3)) + \kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \quad (55)$$

Furthermore, from (50) and (55) we have

$$w(\kappa(\Omega_1,\kappa(\Omega_2,\Omega_3))) = w(\kappa(\Omega_2,\kappa(\Omega_1,\Omega_3))). \quad (56)$$

Hence, the equations (49) and (54) reduces to

$$w(R(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) + w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)) \quad (57)$$

and

$$w(R^*(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3) - w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)), \quad (58)$$

respectively. Also, the equations (50) and (55) reduces to

$$h((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\zeta) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\zeta),\Omega_3) = w((\hat{D}_{\Omega_1}\kappa)(\Omega_2,\Omega_3) - (\hat{D}_{\Omega_2}\kappa)(\Omega_1,\Omega_3)). \quad (59)$$

Likewise, let us obtain some equations about the statistical curvature of a PS-statistical manifold (M,Ψ,w,ζ,h).

From (14), (26) and (45) (or from (19), (46) and (51)), we get

$$R^S(\Omega_1,\Omega_2)\zeta = w(\Omega_1)\Omega_2 - w(\Omega_2)\Omega_1 \quad (60)$$

and from (15) and (26) (or from (19), (47) and (52)), we have

$$R^S(\zeta,\Omega_1)\Omega_2 = w(\Omega_2)\Omega_1 - h(\Omega_1,\Omega_2)\zeta + \kappa(\zeta,\kappa(\Omega_1,\Omega_2)) - \kappa(\Omega_1,\kappa(\zeta,\Omega_2)). \quad (61)$$

From (60) (or from (61) and (45)), we get

$$R^S(\zeta,\Omega_1)\zeta = \Omega_1 - w(\Omega_1)\zeta. \quad (62)$$

From Theorem 3.2-(iv) and (60) (or from (26) and (56)), we obtain that

$$w(R^S(\Omega_1,\Omega_2)\Omega_3) = -w(\Omega_1)h(\Omega_2,\Omega_3) + w(\Omega_2)h(\Omega_1,\Omega_3). \quad (63)$$

At the end of this section, let us deal with the Ricci tensor of these curvatures on a PS-statistical manifold.

Let $\{\Lambda_i\}$, $i = 1,2,...,n$, be an orthonormal basis of the tangent space at any point p of the PS-statistical manifold. From (21), we have

$$\begin{aligned} Ric(\Omega_1,\Omega_2) &= \sum_{i=1}^{n} h(R(\Omega_1,\Lambda_i)\Lambda_i,\Omega_2) \quad (64) \\ &= \widehat{Ric}(\Omega_1,\Omega_2) + \sum_{i=1}^{n} h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i) - (\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i) + \kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i)) - \kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2). \end{aligned}$$

From (12), (56) and (64), we get

$$Ric(\Omega_1,\zeta)=(1-n)w(\Omega_1)+\sum_{i=1}^{n}w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)). \tag{65}$$

Also, using the definition of Ricci tensor, from Lemma 3.1-(iv) and (46) we have

$$\begin{aligned}Ric(\Omega_1,\zeta)&=\sum_{i=1}^{n}h(R(\Omega_1,\Lambda_i)\Lambda_i,\zeta)\\&=(1-n)w(\Omega_1)+\sum_{i=1}^{n}h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i).\end{aligned} \tag{66}$$

From (65) and (66), we get

$$\sum_{i=1}^{n}w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i))=\sum_{i=1}^{n}h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i)$$

and this equation is equivalent with the equation (59).

From (12) and (64) (or from the definition of Ricci tensor and (47)), we have

$$Ric(\zeta,\Omega_1)=(1-n)w(\Omega_1)+\sum_{i=1}^{n}h((\hat{D}_{\zeta}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\zeta,\Lambda_i)+\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i))-\kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \tag{67}$$

Similarly, from (24)

$$\begin{aligned}Ric^*(\Omega_1,\Omega_2)&=\sum_{i=1}^{n}h(R^*(\Omega_1,\Lambda_i)\Lambda_i,\Omega_2) \qquad (68)\\&=\widehat{Ric}(\Omega_1,\Omega_2)-\sum_{i=1}^{n}h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)-\kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i))+\kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2).\end{aligned}$$

From (12), (56) and (68), we get

$$Ric^*(\Omega_1,\zeta)=(1-n)w(\Omega_1)-\sum_{i=1}^{n}w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i)). \tag{69}$$

Also, using the definition of Ricci tensor, from Lemma 3.1-(iv) and (51) we have

$$\begin{aligned}Ric^*(\Omega_1,\zeta)&=\sum_{i=1}^{n}h(R^*(\Omega_1,\Lambda_i)\Lambda_i,\zeta)\\&=(1-n)w(\Omega_1)-\sum_{i=1}^{n}h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\zeta)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\zeta),\Lambda_i).\end{aligned} \tag{70}$$

From (69) and (70), we get

$$\sum_{i=1}^{n}w((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i))=\sum_{i=1}^{n}h((\hat{D}_{\Omega_1}\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\Omega_1,\Lambda_i),\Lambda_i)$$

and this equation is equivalent with the equation (59).

From (12) and (68) (or from the definition of Ricci tensor and (52)), we have

$$Ric^*(\zeta,\Omega_1)=(1-n)w(\Omega_1)-\sum_{i=1}^{n}h((\hat{D}_\zeta\kappa)(\Lambda_i,\Lambda_i)-(\hat{D}_{\Lambda_i}\kappa)(\zeta,\Lambda_i)-\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i))+\kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \quad (71)$$

Finally, let us give similar results for Ricci tensor of the curvature R^S on a PS-statistical manifold.

From (26), we have

$$Ric^S(\Omega_1,\Omega_2)=\widehat{Ric}(\Omega_1,\Omega_2)+\sum_{i=1}^{n}h(\kappa(\Omega_1,\kappa(\Lambda_i,\Lambda_i))-\kappa(\Lambda_i,\kappa(\Omega_1,\Lambda_i)),\Omega_2). \quad (72)$$

From Theorem 3.2-(iv) and (60), we get

$$Ric^S(\Omega_1,\zeta)=\sum_{i=1}^{n}h(R^S(\Omega_1,\Lambda_i)\Lambda_i,\zeta)=-\sum_{i=1}^{n}h(R^S(\Omega_1,\Lambda_i)\zeta,\Lambda_i)=(1-n)w(\Omega_1). \quad (73)$$

From (12) and (72), we have

$$Ric^S(\zeta,\Omega_1)=(1-n)w(\Omega_1)+\sum_{i=1}^{n}h(\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i))-\kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1). \quad (74)$$

Since the Ricci tensor of R^S is symmetric, from (73) and (74) we obtain

$$\sum_{i=1}^{n}h(\kappa(\zeta,\kappa(\Lambda_i,\Lambda_i))-\kappa(\Lambda_i,\kappa(\zeta,\Lambda_i)),\Omega_1)=0. \quad (75)$$

Example 4.6. *Let we deal with the manifold $M=\{(x,y,z)\in\mathbb{R}^3,\ z\neq 0\}$ of dimension 3, where (x,y,z) are the standart coordinates in $\mathbb{R}^3$.*

We choose the vector fields $\{\Lambda_1,\Lambda_2,\Lambda_3\}$ as

$$\Lambda_1=e^x\frac{\partial}{\partial y},\ \Lambda_2=e^x\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right),\ \Lambda_3=-\frac{\partial}{\partial x}, \quad (76)$$

which are linearly independent at each point of M.

Let h be the Riemannian metric defined by $h(\Lambda_i,\Lambda_j)=0$, $i\neq j$, $i,j=1,2,3$ and $h(\Lambda_\kappa,\Lambda_\kappa)=1$, $\kappa=1,2,3$.

Let w be the 1-form defined by $w(\Omega_3)=h(\Omega_3,\Lambda_3)$, for any $\Omega_3\in\Gamma(TM^{(1,0)})$.

Let Ψ be the $(1,1)$-tensor field defined by

$$\Psi\Lambda_1=\Lambda_1,\ \Psi\Lambda_2=\Lambda_2,\ \Psi\Lambda_3=0. \quad (77)$$

Using the linearity of Ψ and h, we have $w(\Lambda_3)=1$, $\Psi^2\Omega_3=\Omega_3-w(\Omega_3)\Lambda_3$ and $h(\Psi\Omega_3,\Psi\Omega_5)=h(\Omega_3,\Omega_5)-w(\Omega_3)w(\Omega_5)$, for any $\Omega_3,\Omega_5\in\Gamma(TM^{(1,0)})$. Thus, for $\Lambda_3=\zeta$, (Ψ,ζ,w,h) defines an almost paracontact metric structure on M.

Now, we have

$$[\Lambda_1,\Lambda_2]=0,\ [\Lambda_1,\Lambda_3]=\Lambda_1,\ [\Lambda_2,\Lambda_3]=\Lambda_2. \quad (78)$$

The L-C connection $\hat{D}$ of h is given by Koszul's formula which is defined as

$$\begin{aligned}2g(\hat{D}_{\Omega_1}\Omega_2,\Omega_3)&=\Omega_1h(\Omega_2,\Omega_3)+\Omega_2h(\Omega_1,\Omega_3)-\Omega_3h(\Omega_1,\Omega_2)\\&\quad-h(\Omega_1,[\Omega_2,\Omega_3])-h(\Omega_2,[\Omega_1,\Omega_3])+h(\Omega_3,[\Omega_1,\Omega_2]).\end{aligned} \quad (79)$$

Taking $\Lambda_3 = \zeta$ and using (79), we have

$$\begin{aligned} &\hat{D}_{\Lambda_1}\Lambda_1 = -\Lambda_3,\ \hat{D}_{\Lambda_1}\Lambda_2 = 0,\ \hat{D}_{\Lambda_1}\Lambda_3 = \Lambda_1,\\ &\hat{D}_{\Lambda_2}\Lambda_1 = 0,\ \hat{D}_{\Lambda_2}\Lambda_2 = -\Lambda_3,\ \hat{D}_{\Lambda_2}\Lambda_3 = \Lambda_2,\\ &\hat{D}_{\Lambda_3}\Lambda_1 = 0,\ \hat{D}_{\Lambda_3}\Lambda_2 = 0,\ \hat{D}_{\Lambda_3}\Lambda_3 = 0. \end{aligned} \tag{80}$$

From above, one can be easily see that (ϕ, ζ, w, h) is a PS-structure on M. Consequently, (M, ϕ, ζ, w, h) is a 3-dimensional PS-manifold (for detail, see [18]).

Now, let us suppose the PS-statistical structure (36) which is defined as $\tilde{D}_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 + w(\Omega_1)w(\Omega_2)\zeta$ ($\tilde{D}^*_{\Omega_1}\Omega_2 = \hat{D}_{\Omega_1}\Omega_2 - w(\Omega_1)w(\Omega_2)\zeta$ and $\kappa(\Omega_1, \Omega_2) = w(\Omega_1)w(\Omega_2)\zeta$) for this para-Sasakian manifold. Thus, from (80) we have

$$\begin{aligned} &\tilde{D}_{\Lambda_1}\Lambda_1 = \tilde{D}^*_{\Lambda_1}\Lambda_1 = -\Lambda_3,\ \tilde{D}_{\Lambda_1}\Lambda_2 = \tilde{D}^*_{\Lambda_1}\Lambda_2 = 0,\ \tilde{D}_{\Lambda_1}\Lambda_3 = \tilde{D}^*_{\Lambda_1}\Lambda_3 = \Lambda_1,\\ &\tilde{D}_{\Lambda_2}\Lambda_1 = \tilde{D}^*_{\Lambda_2}\Lambda_1 = 0,\ \tilde{D}_{\Lambda_2}\Lambda_2 = \tilde{D}^*_{\Lambda_2}\Lambda_2 = -\Lambda_3,\ \tilde{D}_{\Lambda_2}\Lambda_3 = \tilde{D}^*_{\Lambda_2}\Lambda_3 = \Lambda_2,\\ &\tilde{D}_{\Lambda_3}\Lambda_1 = \tilde{D}^*_{\Lambda_3}\Lambda_1 = 0,\ \tilde{D}_{\Lambda_3}\Lambda_2 = \tilde{D}^*_{\Lambda_3}\Lambda_2 = 0,\ \tilde{D}_{\Lambda_3}\Lambda_3 = -\tilde{D}^*_{\Lambda_3}\Lambda_3 = \Lambda_3. \end{aligned} \tag{81}$$

Actually, one can easily see from (81) that, $\tilde{T}(\Lambda_i, \Lambda_j) = 0$ and $(\tilde{D}_{\Lambda_i}h)(\Lambda_j, e_\kappa) = 0$ hold for all $i, j, \kappa = 1, 2, 3$. So, (D, h) is a statistical structure and since $\kappa(\Lambda_i, \Psi e_j) + \Psi\kappa(\Lambda_i, \Lambda_j) = 0$ holds for all $i, j = 1, 2, 3$, $(\tilde{D}, h, \Psi, w, \zeta)$ is a PS-statistical structure on M.

From the above results, we can obtain the components of the curvature tensors with respect to the connections D and D^*, respectively, as follows:

$$\begin{aligned} &\tilde{R}(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}(\Lambda_1, \Lambda_2)\Lambda_3 = 0,\\ &\tilde{R}(\Lambda_1, \Lambda_3)\Lambda_1 = 2\Lambda_3,\ \tilde{R}(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}(\Lambda_1, \Lambda_3)\Lambda_3 = 0,\\ &\tilde{R}(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}(\Lambda_2, \Lambda_3)\Lambda_2 = 2\Lambda_3,\ \tilde{R}(\Lambda_2, \Lambda_3)\Lambda_3 = 0. \end{aligned} \tag{82}$$

and

$$\begin{aligned} &\tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}^*(\Lambda_1, \Lambda_2)\Lambda_3 = 0,\\ &\tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^*(\Lambda_1, \Lambda_3)\Lambda_3 = -2\Lambda_1,\\ &\tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^*(\Lambda_2, \Lambda_3)\Lambda_3 = -2\Lambda_2. \end{aligned} \tag{83}$$

With the help of the equations (82) and (83), we get the Ricci tensors of the curvature tensors $\tilde{R}$ and $\tilde{R}^*$, respectively, as follows:

$$\begin{aligned} &\widetilde{Ric}(\Lambda_1, \Lambda_1) = -1,\ \widetilde{Ric}(\Lambda_1, \Lambda_2) = 0,\ \widetilde{Ric}(\Lambda_1, \Lambda_3) = 0,\\ &\widetilde{Ric}(\Lambda_2, \Lambda_1) = 0,\ \widetilde{Ric}(\Lambda_2, \Lambda_2) = -1,\ \widetilde{Ric}(\Lambda_2, \Lambda_3) = 0,\\ &\widetilde{Ric}(\Lambda_3, \Lambda_1) = 0,\ \widetilde{Ric}(\Lambda_3, \Lambda_2) = 0,\ \widetilde{Ric}(\Lambda_3, \Lambda_3) = -4 \end{aligned} \tag{84}$$

and

$$\begin{aligned} &\widetilde{Ric}^*(\Lambda_1, \Lambda_1) = -3,\ \widetilde{Ric}^*(\Lambda_1, \Lambda_2) = 0,\ \widetilde{Ric}^*(\Lambda_1, \Lambda_3) = 0,\\ &\widetilde{Ric}^*(\Lambda_2, \Lambda_1) = 0,\ \widetilde{Ric}^*(\Lambda_2, \Lambda_2) = -3,\ \widetilde{Ric}^*(\Lambda_2, \Lambda_3) = 0,\\ &\widetilde{Ric}^*(\Lambda_3, \Lambda_1) = 0,\ \widetilde{Ric}^*(\Lambda_3, \Lambda_2) = 0,\ \widetilde{Ric}^*(\Lambda_3, \Lambda_3) = 0. \end{aligned} \tag{85}$$

Furthermore, from the definition of the statistical curvature tensor, (82) and (83), we can obtain the components of the statistical curvature tensor as

$$\begin{aligned} &\tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_1 = \Lambda_2,\ \tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_2 = -\Lambda_1,\ \tilde{R}^S(\Lambda_1, \Lambda_2)\Lambda_3 = 0,\\ &\tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_1 = \Lambda_3\ ,\ \tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_2 = 0,\ \tilde{R}^S(\Lambda_1, \Lambda_3)\Lambda_3 = -\Lambda_1,\\ &\tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_1 = 0,\ \tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_2 = \Lambda_3,\ \tilde{R}^S(\Lambda_2, \Lambda_3)\Lambda_3 = -\Lambda_2 \end{aligned} \tag{86}$$

and from (86), we get the Ricci tensors of the statistical curvature tensor as

$$\begin{aligned}
&\widetilde{Ric}^S(\Lambda_1, \Lambda_1) = -2,\ \widetilde{Ric}^S(\Lambda_1, \Lambda_2) = 0,\ \widetilde{Ric}^S(\Lambda_1, \Lambda_3) = 0, \\
&\widetilde{Ric}^S(\Lambda_2, \Lambda_1) = 0,\ \widetilde{Ric}^S(\Lambda_2, \Lambda_2) = -2,\ \widetilde{Ric}^S(\Lambda_2, \Lambda_3) = 0, \\
&\widetilde{Ric}^S(\Lambda_3, \Lambda_1) = 0,\ \widetilde{Ric}^S(\Lambda_3, \Lambda_2) = 0,\ \widetilde{Ric}^S(\Lambda_3, \Lambda_3) = -2.
\end{aligned} \tag{87}$$

5. SOME CHARACTERIZATIONS FOR THESE MANIFOLDS

In this section, we investigate some special curvature conditions for a PS-statistical manifold. For this, firstly we study on Ricci semi-symmetric and Ricci pseudo-symmetric PS-statistical manifolds and after we give some results for ζ-projectively flat, projectively flat and Ψ-projectively semi-symmetric PS-statistical manifolds.

5.1. Ricci Semi-Symmetric and Ricci Pseudo-Symmetric PS-Statistical Manifolds

We know that, if (M, h) is a connected n-dimensional, $n \geq 3$, semi-Riemannian manifold of class C^∞, then for a $(0, k)$-tensor field T on M, $k \geq 1$, the $(0, k+2)$-tensors $\mathcal{R} \cdot T$ and $Q(h, T)$ are defined by

$$\begin{aligned}
(\mathcal{R} \cdot T)(X_1, ..., X_k; \Omega_1, \Omega_2) &= (R(\Omega_1, \Omega_2) \cdot T)(X_1, ..., X_k) \\
&= -T(R(\Omega_1, \Omega_2)X_1, X_2, ..., X_k) \\
&\quad - ... - T(X_1, ..., X_{k-1}, R(\Omega_1, \Omega_2)X_k)
\end{aligned} \tag{88}$$

and

$$\begin{aligned}
Q(h, T)(X_1, ..., X_k; \Omega_1, \Omega_2) &= ((\Omega_1 \wedge_h \Omega_2) \cdot T)(X_1, ..., X_k) \\
&= -T((\Omega_1 \wedge_h \Omega_2)X_1, X_2, ..., X_k) \\
&\quad - ... - T(X_1, ..., X_{k-1}, (\Omega_1 \wedge_h \Omega_2)X_k)
\end{aligned} \tag{89}$$

respectively, for all $X_1, ..., X_k, \Omega_1, \Omega_2 \in \Gamma(TM^{(1,0)})$. Here R is the Riemannian curvature tensor field of M and $\mathcal{R}$ is the Riemannian Christoffel tensor field given by $\mathcal{R}(\Omega_1, \Omega_2, \Omega_3, \Omega_4) = h(R(\Omega_1, \Omega_2)\Omega_3, \Omega_4)$. Also, the endomorphisms are defined by

$$R(\Omega_1, \Omega_2)\Omega_3 = [D_{\Omega_1}, D_{\Omega_2}]\Omega_3 - D_{[\Omega_1, \Omega_2]}\Omega_3 \tag{90}$$

and

$$(\Omega_1 \wedge_h \Omega_2)\Omega_3 = h(\Omega_2, \Omega_3)\Omega_1 - h(\Omega_1, \Omega_3)\Omega_2. \tag{91}$$

So, we can give the following definition for PS-statistical manifolds:

Definition 5.1. *Let M be an n-dimensional PS-statistical manifold. Then, M is called Ricci pseudo-symmetric with respect to R^S if at every point of M the tensor $\mathcal{R}^S \cdot Ric^S$ and $Q(h, Ric^S)$ are linearly dependent. This is equivalent to the fact that the equality*

$$\mathcal{R}^S \cdot Ric^S = L_{Ric^S} Q(h, Ric^S), \tag{92}$$

hold the set $U_{Ric^S} = \{x \in M : Q(h, Ric^S) \neq 0\}$, for some function L_{Ric^S} on U_{Ric^S}, where Ric^S is the Ricci tensor of R^S. Also, if $L_{Ric^S} = 0$ holds in (92), i.e.,

$$\mathcal{R}^S \cdot Ric^S = 0 \tag{93}$$

holds, then M is called Ricci semi-symmetric with respect to R^S.

Firstly let us assume that M is Ricci semi-symmetric with respect to R^S. Then, we can write

$$\begin{aligned}(\mathcal{R}^S \cdot Ric^S)(U,V;\Omega_1,\Omega_2) &= (R^S(\Omega_1,\Omega_2)\cdot Ric^S)(U,V)\\ &= -Ric^S(R^S(\Omega_1,\Omega_2)U,V) - Ric^S(U,R^S(\Omega_1,\Omega_2)V).\end{aligned} \tag{94}$$

Using (93) in (94), we have

$$Ric^S(R^S(\Omega_1,\Omega_2)U,V) + Ric^S(U,R^S(\Omega_1,\Omega_2)V) = 0. \tag{95}$$

Putting $\Omega_2 = V = \zeta$ in (95) and using (61), we get

$$\begin{aligned}&- w(U)Ric^S(\Omega_1,\zeta) + h(\Omega_1,U)Ric^S(\zeta,\zeta) - Ric^S(\kappa(\zeta,\kappa(\Omega_1,U)),\zeta) + Ric^S(\kappa(\Omega_1,\kappa(\zeta,U)),\zeta)\\ &+ w(\Omega_1)Ric^S(U,\zeta) - Ric^S(U,\Omega_1) = 0.\end{aligned} \tag{96}$$

Using (56) and (73) in (96), we have

$$Ric^S(U,\Omega_1) = (1-n)h(U,\Omega_1).$$

Hence, we can state the following Theorem:

Theorem 5.2. *Let M be a PS-statistical manifold with PS-statistical structure (D,h,Ψ,w,ζ). If M is Ricci semi-symmetric with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

Now, let us assume that M is Ricci pseudo-symmetric with respect to R^S. Then, from (92) we can write

$$(R^S(\Omega_1,\Omega_2)\cdot Ric^S)(U,V) = -L_{Ric^S}\left\{Ric^S((\Omega_1\wedge_h\Omega_2)U,V) + Ric^S(U,(\Omega_1\wedge_h\Omega_2)V)\right\}, \tag{97}$$

for all $\Omega_1,\Omega_2,U,V \in \Gamma(TM^{(1,0)})$. Using (91) in (97), we get

$$-Ric^S(R^S(\Omega_1,\Omega_2)U,V) - Ric^S(U,R^S(\Omega_1,\Omega_2)V) = -L_{Ric^S}\left\{\begin{array}{l} Ric^S(\Omega_1,V)h(\Omega_2,U) - Ric^S(\Omega_2,V)h(\Omega_1,U)\\ +Ric^S(U,\Omega_1)h(\Omega_2,V) - Ric^S(U,\Omega_2)h(\Omega_1,V)\end{array}\right\}. \tag{98}$$

Putting $\Omega_2 = V = \zeta$ in (98) and using (56), (61) and (73), we get

$$Ric^S(U,\Omega_1) = (1-n)h(U,\Omega_1).$$

So, we can give the following Theorem:

Theorem 5.3. *Let M be a PS-statistical manifold with PS-statistical structure (D,h,Ψ,w,ζ). If M is Ricci pseudo-symmetric with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

5.2. Projectively Flat and Ψ-Projectively Semi-Symmetric PS-Statistical Manifolds

Let M be an n-dimensional PS-statistical manifold. Then, the projective curvature tensor P^S of M with respect to the statistical curvature R^S is defined by

$$P^S(\Omega_1,\Omega_2)\Omega_3 = R^S(\Omega_1,\Omega_2)\Omega_3 - \frac{1}{n-1}\left\{Ric^S(\Omega_2,\Omega_3)\Omega_1 - Ric^S(\Omega_1,\Omega_3)\Omega_2\right\} \tag{99}$$

for all $\Omega_1,\Omega_2,\Omega_3 \in \Gamma(TM^{(1,0)})$.

Definition 5.4. *A PS-statistical manifold is called projectively flat with respect to the statistical curvature R^S, if the projective curvature tensor P^S vanishes at each point of the manifold. Also, a PS-statistical manifold is called ζ-projectively flat with respect to the statistical curvature R^S, if $P^S(\Omega_1,\Omega_2)\zeta = 0$ holds for all $\Omega_1,\Omega_2 \in \Gamma(TM^{(1,0)})$.*

Theorem 5.5. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). Then, M is ζ-projectively flat with respect to the statistical curvature R^S.*

Proof. It is obvious from (60), (73) and (99). □

Now, let us suppose that M is projectively flat with respect to the statistical curvature R^S. Then, since $P^S = 0$, from (99) we can write

$$R^S(\Omega_1, \Omega_2)\Omega_3 = \frac{1}{n-1}\left\{Ric^S(\Omega_2, \Omega_3)\Omega_1 - Ric^S(\Omega_1, \Omega_3)\Omega_2\right\}. \tag{100}$$

Taking $\Omega_1 = \zeta$ in (100) and using (61), we have

$$w(\Omega_3)\Omega_2 - h(\Omega_2, \Omega_3)\zeta + \kappa(\zeta, \kappa(\Omega_2, \Omega_3)) - \kappa(\Omega_2, \kappa(\zeta, \Omega_3)) = \frac{1}{n-1}\left\{Ric^S(\Omega_2, \Omega_3)\zeta - (1-n)w(\Omega_3)\Omega_2\right\}. \tag{101}$$

Applying w to (101), from (56) we get

$$Ric^S(\Omega_2, \Omega_3) = (1-n)h(\Omega_2, \Omega_3).$$

Thus, we have

Theorem 5.6. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). If M is projectively flat with respect to statistical curvature R^S, then M is Einstein with respect to Ricci tensor of R^S.*

Definition 5.7. *A PS-statistical manifold is called Ψ-projectively semi-symmetric with respect to the statistical curvature R^S, if it satisfies $(P^S(\Omega_1, \Omega_2)\Psi)\Omega_3 = 0$ holds for all $\Omega_1, \Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$.*

Finally, let us assume that M is Ψ-projectively semi-symmetric with respect to the statistical curvature R^S. So, from $(P^S(\Omega_1, \Omega_2)\Psi)\Omega_3 = 0$ we can write

$$P^S(\Omega_1, \Omega_2)\Psi\Omega_3 - \Psi P^S(\Omega_1, \Omega_2)\Omega_3 = 0. \tag{102}$$

Using (99) in (102) and taking $\Omega_1 = \zeta$, from (61) and (74) we have

$$-h(\Omega_2, \Psi\Omega_3)\zeta + \kappa(\zeta, \kappa(\Omega_2, \Psi\Omega_3)) - \kappa(\Omega_2, \kappa(\zeta, \Psi\Omega_3)) - \frac{1}{n-1}Ric^S(\Omega_2, \Psi\Omega_3)\zeta - \Psi\kappa(\zeta, \kappa(\Omega_2, \Omega_3)) + \Psi\kappa(\Omega_2, \kappa(\zeta, \Omega_3)) = 0. \tag{103}$$

Applying w to (103), from (56) we get

$$Ric^S(\Omega_2, \Psi\Omega_3) = (1-n)h(\Omega_2, \Psi\Omega_3). \tag{104}$$

Taking $\Psi\Omega_2$ instead of Ω_2 and using (9), (29), (56) and (72), we obtain

$$Ric^S(\Omega_2, \Omega_3) + 2\sum_{i=1}^{n}\{-h(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)), \Omega_3) + w(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)))w(\Omega_3)\} = (1-n)h(\Omega_2, \Omega_3). \tag{105}$$

Hence, we can give the following Theorem:

Theorem 5.8. *Let M be a PS-statistical manifold with PS-statistical structure (D, h, Ψ, w, ζ). If M is Ψ-projectively semi-symmetric with respect to statistical curvature R^S and*

$$\sum_{i=1}^{n}\{h(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)), \Omega_3) - w(\kappa(\Omega_2, \kappa(\Lambda_i, \Lambda_i)))w(\Omega_3)\} = 0$$

holds for all $\Omega_2, \Omega_3 \in \Gamma(TM^{(1,0)})$, then M is Einstein with respect to Ricci tensor of R^S.

6. CONCLUSION

One of the fundamental concept in information theory is that of the Fisher-Rao Information Matrix, which provides us with another measure of the distance between two different probability distributions. Such a measure endows the statistical manifold with a Riemannian structure. In fact, while the relative entropy does not define a real distance between distributions (for example, it is not symmetric), it can be shown that the Fisher-Rao Information Matrix arises as the Hessian of the relative entropy over a stationary point. The entries of such a matrix are in correspondence with the components of the metric tensor over the manifold of probability distributions [4].

On the other hand, the role played by differential geometry in statistics was not fully acknowledged until 1975 when Efron first introduced the concept of statistical curvature for one-parameter models and emphasized its importance in the theory of statistical estimation. Efron pointed out how any regular parametric family could be approximated locally by a curved exponential family and that the curvature of these models measures their departure from exponentiality. It turned out that this concept was intimately related to Fisher's theory of information loss. Efron's formal theory did not use all the bells and whistles of differential geometry. The first step to an elegant geometric theory was done by Dawid, who introduced a connection on the space of all positive probability distributions and showed that Efron's statistical curvature is induced by this connection. The use of differential geometry in its elegant splendor for the elaboration of previous ideas was systematically achieved by Amari, who studied the informational geometric properties of a manifold with a Fisher metric on it. This is the reason why sometimes this is also called the Fisher–Efron–Amari theory [5]. In the light of these studies, we focused on the curvature tensors of para-Sasakian (PS) statistical manifolds in terms of differential geometry. We started building this with Theorem 1, which we have used connections while doing it. In the context of PS geometry there is another connection of geometric significance which is parallel with respect to the metric and the other tensors defining the contact-metric strucuture. We have given our results using the connections ∇ and ∇^* on statistical manifolds. We have also studied the Ricci tensor of the statistical curvature and studied the cases of the manifold being Einstein under certain conditions in Theorem 3, Theorem 4 and Theorem 6. We have proved the projective flatness of the PS-statistical manifold. We believe that the concepts investigated in this work can be also studied in some new settings. The submanifolds of this subject can be examined as well as the inequality situation.

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