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Bi-Periodic Generalized Fibonacci Polynomials

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Abstract. In this paper, we define bi-periodic generalized Fibonacci polynomials, which generalize Fibonacci, Pell, Jacobsthal, Fermat, Chebyshev polynomials and the other well-known polynomials. We obtain generating functions, Binet formulas and some properties of these polynomials. Also, we prove some fundamental identities conform to the known results of Fibonacci polynomials.

1. Introduction

Polynomials in many fields of mathematics and science are emerged as the generalizations of numbers. Fibonacci polynomials, one of the special polynomials in the literature, are a generalization of well-known Fibonacci numbers defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ with initial terms $f_0 = 0$, $f_1 = 1$ [1]. The nth Fibonacci polynomial $f_n(x)$, is defined by the recurrence relation

$$f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \quad n \geq 2$$

with initial terms $f_0(x) = 0$, $f_1(x) = 1$, and terms of the sequence $\left\{0, 1, \ x, x^2+1, x^3+2x, x^4+3x^2+1, \ldots\right\}$ are Fibonacci polynomials. Many polynomials related to numbers defined by the recurrence relations have been presented in different ways as generalizations of the Fibonacci polynomials called generalized Fibonacci and generalized Fibonaci type polynomials. One of the ways of generalization is to add integers or variables to the recurrence relation of the Fibonacci polynomials. For instance, Pell polynomials are defined by the recurrence relation $p_n(x) = 2xp_{n-1}(x) + p_{n-2}(x)$ with initial terms $p_0(x) = 0$, $p_1(x) = 1$ for $n \geq 2$. Then Jacobsthal polynomials are defined by the recurrence relation $j_n(x) = j_{n-1}(x) + 2x j_{n-2}(x)$ with initial terms $J_0(x) = 0$, $J_1(x) = 1$ for $n \geq 2$ [2, 3]. For the parameter variables x and y in the recurrence relation, bivariate Fibonacci polynomials are introduced by the recurrence relation

$$f_n(x,y) = x f_{n-1}(x,y) + y f_{n-2}(x,y), \quad f_n(x,y) = 0, \ f_1(x,y) = 1, \quad n \geq 2$$

where $x, y \neq 0$, $x^2 + 4y \neq 0$ and generalized identities of these polynomials are obtained [4, 5]. Then, $h(x)$-Fibonacci polynomials as another generalization of Fibonacci polynomials are defined by the recurrence relation

$$f_{h,n}(x) = h(x) f_{h,n-1}(x) + f_{h,n-2}(x), \quad f_{h,n}(x) = 0, \ f_{h,1}(x) = 1, \quad n \geq 2$$

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where $h(x)$ be a polynomial with real coefficients [6]. Further generalizations of Fibonacci polynomials have been presented by many authors as Fermat, Chebyshev, Morgan-Voyce, Vieta polynomials. The generating functions, exponential generating functions, the Binet-like formulas, sums formulas, matrix representations and periods according to the m modulo of Fibonacci polynomial sequences are presented [7–10].

Motivated by of the above-cited studies, it is introduced a new generalization of the Fibonacci numbers and polynomials called generalized Fibonacci polynomials. For $n \geq 2$, the generalized Fibonacci polynomial sequences, $\{\mathcal{F}_n(x)\}_{n\geq 0}$ are defined by the recurrence relation

$$\mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x) \tag{1}$$

with initial terms $\mathcal{F}_0(x) = 0$ and $\mathcal{F}_1(x) = 1$ where $d(x)$ and $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$ [11]. Obviously, for $d(x) = x$ and $g(x) = 1$ we obtain classical Fibonacci polynomial and $\mathcal{F}_n(1) = f_n$ where f_n is the nth classical Fibonacci number. Binet formulas for the generalized Fibonacci polynomial sequences are given by

$$\mathcal{F}_n(x) = \frac{\sigma^n(x) - \rho^n(x)}{\sigma(x) - \rho(x)}$$

where $\sigma(x)$ and $\rho(x)$ are the roots of the quadratic equation $t^2 - d(x)t - g(x) = 0$ of equation (1). The readers can find more detailed information about the generalized Fibonacci polynomial in [12, 13].

In other generalizations of Fibonacci numbers and polynomials, nonzero real numbers are taken into account, bi-periodic Fibonacci number sequences, $\{q_n\}$ are defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial terms $q_0 = 0$, $q_1 = 1$ [14] and bi-periodic Fibonacci polynomial sequences, $\{q_n(x)\}$ are defined by

$$q_n(x) = \begin{cases} aq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is even} \\ bq_{n-1}(x) + q_{n-2}(x), & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial terms $q_0(x) = 0$, $q_1(x) = 1$ where a and b are any two nonzero real numbers. Also, some identities related to these bi-periodic sequences are given, respectively [15].

The aim of this study is to define new generalizations of the Fibonaci and the Fibonacci type polynomials, the bi-periodic Fibonacci and the bi-periodic Fibonacci type polynomials, which we shall call bi-periodic generalized Fibonacci polynomials. It is to present generating functions, general formulas and well-known identities for these polynomials. It is also to give special cases of the bi-periodic generalized Fibonacci polynomials and generalize all the results.

2. Bi-Periodic Generalized Fibonacci Polynomials

In this section we define a new kind of generalized Fibonacci polynomials, called bi-periodic generalized Fibonacci polynomials, which are Fibonacci polynomials, $h(x)$-Fibonacci polynomials, Fibonacci polynomials with two variables, Pell polynomials, Jacobsthal polynomials, Fermat polynomials, Chebyshev second kind polynomials, Morgan-Voyce first kind polynomials and Vieta polynomials. Generating functions, Binet formulas, some basic properties as well as the Catalan's identity, Cassini's identity, d'Ocagne's identity for these polynomials are obtained.

Definition 2.1. *For any two nonzero real numbers a and b, the nth bi-periodic generalized Fibonacci polynomial is defined by the recurrence relation*

$$\mathbb{F}_n(x) = \begin{cases} ad(x)\mathbb{F}_{n-1}(x) + g(x)\mathbb{F}_{n-2}(x), & \text{if } n \text{ is even} \\ bd(x)\mathbb{F}_{n-1}(x) + g(x)\mathbb{F}_{n-2}(x), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \tag{2}$$

with initial terms $\mathbb{F}_0(x) = 0$, $\mathbb{F}_1(x) = 1$ *for* $n \geq 2$, *where* $d(x)$ *and* $g(x)$ *are fixed nonzero polynomials in* $\mathbb{Q}[x]$. *The bi-periodic generalized Fibonacci polynomial sequences are denoted by* $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$.

The bi-periodic generalized Fibonacci polynomial sequences are as follows

$$\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}} = \Big\{0, 1, ad(x), abd^2(x) + g(x), a^2bd^3(x) + 2ad(x)g(x), \ a^2b^2d^4(x) + 3abd^2(x)g(x) + g^2(x), \\ a^3b^2d^5(x) + 4a^2bd^3(x)g(x) + 3ad(x)g^2(x), a^3b^3d^6(x) + 5a^2b^2d^4(x)g(x) + 6abd^2(x)g^2(x) + g^3(x), \ldots\Big\}$$

Note that $d(x) = x$ and $g(x) = 1$, we get the bi-periodic Fibonacci polynomial $\mathbb{F}_n(x) = F_n(x)$. Similar special cases of the bi-periodic generalized Fibonacci polynomials are given in the Table 1

Table 1: Special cases of the polynomials $\mathbb{F}_n(x)$

Bi-Periodic Generalized Fibonacci Polynomials	$\mathbb{F}_n$	$d(x)$	$g(x)$
Bi-periodic Fibonacci polynomials	$F_n(x)$	x	1
Bi-periodic $h(x)$-Fibonacci polynomials	$F_{h,n}(x)$	$h(x)$	1
Bi-periodic Fibonacci polynomials with two variables	$F_n(x, y)$	x	y
Bi-periodic Pell polynomials	$P_n(x)$	$2x$	1
Bi-periodic Jacobsthal polynomials	$J_n(x)$	1	$2x$
Bi-periodic Fermat polynomials	$\Phi_n(x)$	$3x$	-2
Bi-periodic Chebyshev second kind polynomials	$U_n(x)$	$2x$	-1
Bi-periodic Morgan-Voyce first kind polynomials	$B_n(x)$	$x+2$	-1
Bi-periodic Vieta polynomials	$V_n(x)$	x	-1

Since the all results given throughout the study are provided for all the bi-periodic generalized Fibonacci polynomials, the values given in Table 1 can be used in the relevant theorem or corollary for any bi-periodic polynomials.

From Definition 2.1, alternative recurrence relations can be given for the bi-periodic generalized Fibonacci polynomials where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e.,

$$\xi(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}.$$

Let a and b be any two nonzero real numbers, nth bi-periodic generalized Fibonacci polynomial is given by

$$\mathbb{F}_n(x) = a^{1-\xi(n)}b^{\xi(n)}d(x)\mathbb{F}_{n-1}(x) + g(x)\mathbb{F}_{n-2}(x), \quad n \geq 2 \tag{3}$$

with initial terms $\mathbb{F}_0(x) = 0$, $\mathbb{F}_1(x) = 1$ where $d(x)$ and $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$.

The quadratic equation of the bi-periodic generalized Fibonacci polynomials is

$$t^2 - d(x)abt - g(x)ab = 0$$

and their roots are $\gamma(x) = \frac{d(x)ab + \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$ and $\delta(x) = \frac{d(x)ab - \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$. In this case, the following relations are obtained between the roots $\gamma(x)$ and $\delta(x)$

$$\gamma(x) + \delta(x) = d(x)\,ab$$

$$\gamma(x) - \delta(x) = \sqrt{d^2(x)\,a^2b^2 + 4g(x)\,ab}$$

$$\gamma(x)\,\delta(x) = -g(x)\,ab$$

$$d(x)\,\gamma(x) + g(x) = \frac{\gamma^2(x)}{ab}$$

$$d(x)\,\delta(x) + g(x) = \frac{\delta^2(x)}{ab}.$$

2.1. Generating Functions and Binet Formulas of Polynomials $\mathbb{F}_n(x)$

In this section, we construct the generating functions of the bi-periodic generalized Fibonacci polynomial the sequences, $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$. Let the generating functions of these sequences be $G_n(x,t)$ such that

$$G_n(x,t) = \sum_{n=0}^{\infty} \mathbb{F}_n(x)\,t^n \tag{4}$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial and $d(x)$, $g(x)$ are fixed nonzero polynomials in $\mathbb{Q}[x]$. First, the identities for the odd and even subscript terms of the bi-periodic generalized Fibonacci polynomials are given in the following lemma used to derive these functions.

Lemma 2.2. *The bi-periodic generalized Fibonacci polynomial sequences, $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$ satisfy the following identities*

i. $\mathbb{F}_{2n}(x) = \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)$

ii. $\mathbb{F}_{2n+1}(x) = \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)$

Proof. Using the equation (2)

i.

$$\begin{aligned}
\mathbb{F}_{2n}(x) &= ad(x)\,\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-2}(x)\\
&= ad(x)\,(bd(x)\,\mathbb{F}_{2n-2}(x) + g(x)\,\mathbb{F}_{2n-3}(x)) + g(x)\,\mathbb{F}_{2n-2}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-2}(x) + ad(x)\,g(x)\,\mathbb{F}_{2n-3}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-2}(x) + g(x)\,\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)\\
&= \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-2}(x) - g^2(x)\,\mathbb{F}_{2n-4}(x)
\end{aligned}$$

ii.

$$\begin{aligned}
\mathbb{F}_{2n+1}(x) &= bd(x)\,\mathbb{F}_{2n}(x) + g(x)\,\mathbb{F}_{2n-1}(x)\\
&= bd(x)\,(ad(x)\,\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-2}(x)) + g(x)\,\mathbb{F}_{2n-1}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-1}(x) + bd(x)\,g(x)\,\mathbb{F}_{2n-2}(x)\\
&= \left(abd^2(x) + g(x)\right)\mathbb{F}_{2n-1}(x) + g(x)\,\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)\\
&= \left(abd^2(x) + 2g(x)\right)\mathbb{F}_{2n-1}(x) - g^2(x)\,\mathbb{F}_{2n-3}(x)
\end{aligned}$$

Thus, the proof is completed.

Using the Lemma 2.2, the generating functions of the sequences $\{\mathbb{F}_n(x)\}_{n\in\mathbb{N}}$ are given in the following Theorem.

Theorem 2.3. *The generating functions for the bi-periodic generalized Fibonacci polynomial sequences are*

$$G_n(x,t)=\frac{t+ad(x)t^2-g(x)t^3}{1-(abd^2(x)+2g(x))t^2+g^2(x)t^4}.$$

Proof. Using equation 4, we get

$$G_n(x,t)=\sum_{n=0}^{\infty}\mathbb{F}_n(x)t^n=\mathbb{F}_0(x)+\mathbb{F}_1(x)t+\mathbb{F}_2(x)t^2+\ldots+\mathbb{F}_n(x)t^n+\ldots$$

Let generating functions $G_n(x,t)$ be the sum of the odd subscript and even subscript terms separately. Then

$$G_n(x,t)=G_n^{Ç}(x,t)+G_n^{T}(x,t) \tag{5}$$

where $G_n^{Ç}(x,t)$ is the sum of the even subscript terms and $G_n^{T}(x,t)$ is the sum of the odd subscript terms. Therefore,

$$G_n^{Ç}(x,t)=\sum_{i=0}^{\infty}\mathbb{F}_{2i}(x)t^{2i}=\mathbb{F}_0(x)+\mathbb{F}_2(x)t^2+\mathbb{F}_4(x)t^4+\ldots \tag{6}$$

If both sides of equation (6) are multiplied by $-\left(abd^2(x)+2g(x)\right)t^2$ and $g^2(x)t^4$, then we get

$$-\left(abd^2(x)+2g(x)\right)t^2G_n^{Ç}(x,t)=-abd^2(x)+2g(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i}(x)t^{2i+2} \tag{7}$$

and

$$g^2(x)t^4G_n^{Ç}(x,t)=g^2(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i}(x)t^{2i+4} \tag{8}$$

If we add the equations (6), (7) and (8) side by side, we obtain

$$\begin{aligned}\left(1-\left(abd^2(x)+2g(x)\right)t^2+g^2(x)t^4\right)G_n^{Ç}(x,t)&=\mathbb{F}_0(x)+\mathbb{F}_2(x)t^2+\sum_{i=2}^{\infty}\mathbb{F}_{2i}(x)t^{2i}\\&-\left(abd^2(x)+2g(x)\right)\sum_{i=0}^{\infty}\mathbb{F}_{2i}(x)t^{2i+2}+g^2(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i}(x)t^{2i+4}\\&=ad(x)t^2+\sum_{i=2}^{\infty}\mathbb{F}_{2i}(x)t^{2i}-\left(abd^2(x)+2g(x)\right)\sum_{i=2}^{\infty}\mathbb{F}_{2i-2}(x)t^{2i}\\&+g^2(x)\sum_{i=2}^{\infty}\mathbb{F}_{2i-4}(x)t^{2i}\\&=ad(x)t^2+\sum_{i=2}^{\infty}\left(\mathbb{F}_{2i}(x)-\left(abd^2(x)+2g(x)\right)\mathbb{F}_{2i-2}(x)+g^2(x)\mathbb{F}_{2i-4}(x)\right)t^{2i}.\end{aligned}$$

Using Lemma 2.2, i., generating functions for even subscript terms in the bi-periodic generalized Fibonacci polynomial sequences are obtained as

$$G_n^{Ç}(x,t)=\frac{ad(x)t^2}{1-(abd^2(x)+2g(x))t^2+g^2(x)t^4}.$$

Now let consider the sum of the odd subscript terms in the generating function. Therefore,

$$G_n^T(x,t)=\sum_{i=0}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+1}=\mathbb{F}_1(x)\,t+\mathbb{F}_3(x)\,t^3+\mathbb{F}_5(x)\,t^5+\ldots \tag{9}$$

If both sides of equation (9) are multiplied by $-\left(abd^2(x)+2g(x)\right)t^2$ and $g^2(x)\,t^4$, , then we get

$$-\left(abd^2(x)+2g(x)\right)t^2G_n^T(x,t)=-\left(abd^2(x)+2g(x)\right)\sum_{i=0}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+3} \tag{10}$$

and

$$g^2(x)\,t^4G_n^T(x,t)=g^2(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+5} \tag{11}$$

If we add the equations (9), (10) and (11) side by side, we obtain

$$\begin{aligned}\left(1-\left(abd^2(x)+2g(x)\right)t^2+g^2(x)\,t^4\right)G_n^T(x,t)&=\mathbb{F}_1(x)\,t+\mathbb{F}_3(x)\,t^3+\sum_{i=2}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+1}\\&-\left(abd^2(x)+2g(x)\right)\mathbb{F}_1(x)\,t^3-\left(abd^2(x)+2g(x)\right)\sum_{i=1}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+3}\\&+g^2(x)\sum_{i=0}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+5}\\&=t+\left(abd^2(x)+g(x)\right)t^3+\sum_{i=2}^{\infty}\mathbb{F}_{2i+1}(x)\,t^{2i+1}-\left(abd^2(x)+2g(x)\right)t^3\\&-\left(abd^2(x)+2g(x)\right)\sum_{i=2}^{\infty}\mathbb{F}_{2i-1}(x)\,t^{2i+1}+g^2(x)\sum_{i=2}^{\infty}\mathbb{F}_{2i-3}(x)\,t^{2i+1}\\&=t+\left(abd^2(x)+g(x)\right)t^3-\left(abd^2(x)+2g(x)\right)t^3\\&+\sum_{i=2}^{\infty}\left(\mathbb{F}_{2i+1}(x)-\left(abd^2(x)+2g(x)\right)\mathbb{F}_{2i-1}(x)+g^2(x)\,\mathbb{F}_{2i-3}(x)\right)t^{2i+1}.\end{aligned}$$

Using Lemma 2.2, ii., generating functions for even subscript terms in the bi-periodic generalized Fibonacci polynomial sequences are obtained as

$$G_n^T(x,t)=\frac{t-g(x)\,t^3}{1-\left(abd^2(x)+2g(x)\right)t^2+g^2(x)\,t^4}$$

From equation (5), generating functions for the bi-periodic generalized Fibonacci polynomial sequences are

$$G_n(x,t)=\frac{t+ad(x)\,t^2-g(x)\,t^3}{1-\left(abd^2(x)+2g(x)\right)t^2+g^2(x)\,t^4}.$$

Thus, the proof is completed.

Now we give Binet formulas that allow us to calculate the nth terms of sequences $\{\mathbb{F}_n(x)\}_{n\epsilon\mathbb{N}}$ in the following theorem.

Theorem 2.4. *The Binet formulas for the bi-periodic generalized Fibonacci polynomial sequences are given by*

$$\mathbb{F}_n(x) = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}$$

where $\gamma(x) = \frac{d(x)ab + \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$, $\delta(x) = \frac{d(x)ab - \sqrt{d^2(x)a^2b^2 + 4g(x)ab}}{2}$ *and* $\xi(n) = n - 2\left[\frac{n}{2}\right]$.

Proof. By induction method on n. The result is obviously valid for $n = 0, 1$. Suppose that result is true for $n \in \mathbb{N}$, we shall show that it is true for $n+1$. Using equation (3) and the hypothesis of induction, we have

$$\begin{aligned}
\mathbb{F}_{n+1}(x) &= a^{1-\xi(n+1)}b^{\xi(n+1)}d(x)\,\mathbb{F}_n(x) + g(x)\,\mathbb{F}_{n-1}(x) \\
&= a^{1-\xi(n+1)}b^{\xi(n+1)}d(x)\left(\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}\right) + g(x)\left(\left(\frac{a^{1-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\right)\frac{\gamma^{n-1}(x) - \delta^{n-1}(x)}{\gamma(x) - \delta(x)}\right) \\
&= \frac{a^{1-\xi(n+1)}\gamma^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{a^{1-\xi(n)}b^{\xi(n+1)}d(x)\gamma(x)}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{1-\xi(n-1)}g(x)}{a^{1-\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\right) \\
&- \frac{a^{1-\xi(n+1)}\delta^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{a^{1-\xi(n)}b^{\xi(n+1)}d(x)\delta(x)}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{1-\xi(n-1)}g(x)}{a^{1-\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor}}\right) \\
&= \frac{a^{1-\xi(n+1)}\gamma^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{abd(x)\gamma(x)}{a^{\xi(n)}b^{1-\xi(n+1)}(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{abg(x)}{(ab)^{\lfloor \frac{n-1}{2} \rfloor + 1}}\right) \\
&- \frac{a^{1-\xi(n+1)}\delta^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{abd(x)\delta(x)}{a^{\xi(n)}b^{1-\xi(n+1)}(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{abg(x)}{(ab)^{\lfloor \frac{n-1}{2} \rfloor + 1}}\right) \\
&= \frac{a^{1-\xi(n+1)}\gamma^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{ab(d(x)\gamma(x) + g(x))}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) - \frac{a^{1-\xi(n+1)}\delta^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{ab(d(x)\delta(x) + g(x))}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) \\
&= \frac{a^{1-\xi(n+1)}\gamma^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{\gamma^2(x)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) - \frac{a^{1-\xi(n+1)}\delta^{n-1}(x)}{\gamma(x) - \delta(x)}\left(\frac{\delta^2(x)}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) \\
&= \left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right)\frac{\gamma^{n+1}(x) - \delta^{n+1}(x)}{\gamma(x) - \delta(x)}
\end{aligned}$$

where $d(x)\gamma(x) + g(x) = \frac{\gamma^2(x)}{ab}$, $d(x)\delta(x) + g(x) = \frac{\delta^2(x)}{ab}$ and $\xi(n) + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$, $1 - \xi(n+1) + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$. This completes the proof.

2.2. *Identities for Polynomials $\mathbb{F}_n(x)$*

In this section, we give various identities for consecutive terms and negative subscript terms of the bi-periodic generalized Fibonacci polynomial sequences and present the Catalan's identity, Cassini's identity, d'Ocagne's identity for these polynomials.

Theorem 2.5. *The limit of the ratio of consecutive terms of the bi-periodic generalized Fibonacci polynomial sequences is*

i. $lim_{n\to\infty}\frac{\mathbb{F}_{2n+1}(x)}{\mathbb{F}_{2n}(x)} = \frac{\gamma(x)}{a}$

ii. $lim_{n\to\infty}\frac{\mathbb{F}_{2n}(x)}{\mathbb{F}_{2n-1}(x)} = \frac{\gamma(x)}{b}$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial.

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

i.

$$
\begin{aligned}
lim_{n\to\infty}\frac{\mathbb{F}_{2n+1}(x)}{\mathbb{F}_{2n}(x)} &= lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n+1)}}{(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\right)\left(\frac{\gamma^{2n+1}(x)-\delta^{2n+1}(x)}{\gamma(x)-\delta(x)}\right)}{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\lfloor\frac{2n}{2}\rfloor}}\right)\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)} \\
&= lim_{n\to\infty}\frac{\frac{1}{(ab)^n}\left(\frac{\gamma^{2n+1}(x)-\delta^{2n+1}(x)}{\gamma(x)-\delta(x)}\right)}{\frac{a}{(ab)^n}\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)} \\
&= lim_{n\to\infty}\frac{1}{a}\frac{\gamma^{2n+1}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n+1}\right)}{\gamma^{2n}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n}\right)} \\
&= \frac{\gamma(x)}{a}
\end{aligned}
$$

ii.

$$
\begin{aligned}
lim_{n\to\infty}\frac{\mathbb{F}_{2n}(x)}{\mathbb{F}_{2n-1}(x)} &= lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\lfloor\frac{2n}{2}\rfloor}}\right)\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}{\left(\frac{a^{1-\xi(2n-1)}}{(ab)^{\lfloor\frac{2n-1}{2}\rfloor}}\right)\left(\frac{\gamma^{2n-1}(x)-\delta^{2n-1}(x)}{\gamma(x)-\delta(x)}\right)} \\
&= lim_{n\to\infty}\frac{\frac{a}{(ab)^n}\left(\frac{\gamma^{2n}(x)-\delta^{2n}(x)}{\gamma(x)-\delta(x)}\right)}{\frac{1}{(ab)^{n-1}}\left(\frac{\gamma^{2n-1}(x)-\delta^{2n-1}(x)}{\gamma(x)-\delta(x)}\right)} \\
&= lim_{n\to\infty}\frac{a}{ab}\frac{\gamma^{2n}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n}\right)}{\gamma^{2n-1}(x)\left(1-\left(\frac{\delta(x)}{\gamma(x)}\right)^{2n-1}\right)} \\
&= \frac{\gamma(x)}{b}
\end{aligned}
$$

where $|\delta(x)| < \gamma(x)$ and $lim_{n\to\infty}\left(\frac{\delta(x)}{\gamma(x)}\right)^n = 0$. This completes the proof.

Theorem 2.6. *Negative subscript terms of the bi-periodic generalized Fibonacci polynomial sequences are obtained as*

$$\mathbb{F}_{-n}(x) = (-1)^{n+1}(g(x))^{-n}\,\mathbb{F}_n(x).$$

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$
\begin{aligned}
\mathbb{F}_{-n}(x) &= \left(\frac{a^{1-\xi(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}}\right)\frac{\gamma^{-n}(x)-\delta^{-n}(x)}{\gamma(x)-\delta(x)} \\
&= (-1)\left(\frac{a^{1-\xi(-n)}}{(ab)^{\lfloor\frac{-n}{2}\rfloor}}\right)\frac{\gamma^{n}(x)-\delta^{n}(x)}{(-g(x)\,ab)^n(\gamma(x)-\delta(x))} \\
&= (-1)(-g(x))^{-n}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)} \\
&= (-1)^{n+1}(g(x))^{-n}\,\mathbb{F}_n(x)
\end{aligned}
$$

where $\gamma(x)\delta(x) = -g(x)ab$. Thus, the proof is completed.

Now we present some basic identities for the bi-periodic generalized Fibonacci polynomials, such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 2.7. *(Catalan's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have*

$$a^{\xi(n-r)}b^{1-\xi(n-r)}\mathbb{F}_{n-r}(x)\mathbb{F}_{n+r}(x) - a^{\xi(n)}b^{1-\xi(n)}\mathbb{F}_n{}^2(x) = -(-g(x))^{n-r}a^{\xi(r)}b^{1-\xi(r)}\mathbb{F}_r{}^2(x)$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial.

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$\begin{aligned}
&a^{\xi(n-r)}b^{1-\xi(n-r)}\mathbb{F}_{n-r}(x)\mathbb{F}_{n+r}(x) - a^{\xi(n)}b^{1-\xi(n)}\mathbb{F}_n{}^2(x)\\
&= a^{\xi(n-r)}b^{1-\xi(n-r)}\left(\frac{a^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(n+r)}}{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}\right)\left(\frac{\gamma^{n-r}(x)-\delta^{n-r}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{n+r}(x)-\delta^{n+r}(x)}{\gamma(x)-\delta(x)}\right)\\
&- a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{n}(x)-\delta^{n}(x)}{\gamma(x)-\delta(x)}\right)\\
&= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor+\lfloor\frac{n+r}{2}\rfloor}}\left(\frac{\gamma^{2n}(x)-\gamma^{n-r}(x)\delta^{n+r}(x)-\delta^{n-r}(x)\gamma^{n+r}(x)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\frac{\gamma^{2n}(x)-2\gamma^{n}(x)\delta^{n}(x)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\left(\frac{\gamma^{2n}(x)-(\gamma(x)\delta(x))^{n-r}\left(\gamma^{2r}(x)+\delta^{2r}(x)\right)+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{n-\xi(n)}}\left(\frac{\gamma^{2n}(x)-2(\gamma(x)\delta(x))^{n}+\delta^{2n}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a}{(ab)^{n-1}}\left(\frac{-(\gamma(x)\delta(x))^{n-r}\left(\gamma^{2r}(x)+\delta^{2r}(x)\right)+2(\gamma(x)\delta(x))^{n}}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{-a(\gamma(x)\delta(x))^{n-r}}{(ab)^{n-1}}\left(\frac{\gamma^{r}(x)-\delta^{r}(x)}{\gamma(x)-\delta(x)}\right)^2\\
&= \frac{-a(-g(x)ab)^{n-r}}{(ab)^{n-1}}\frac{(ab)^{2\lfloor\frac{r}{2}\rfloor}}{a^{2-2\xi(r)}}\mathbb{F}_r{}^2(x)\\
&= -(-g(x))^{n-r}\frac{a(ab)^{2\lfloor\frac{r}{2}\rfloor}}{(ab)^{\xi(r)+2\lfloor\frac{r}{2}\rfloor-1}a^{2-2\xi(r)}}\mathbb{F}_r{}^2(x)\\
&= -(-g(x))^{n-r}a^{\xi(r)}b^{1-\xi(r)}\mathbb{F}_r{}^2(x)
\end{aligned}$$

where $\xi(n) = n - 2\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lfloor\frac{n-r}{2}\right\rfloor + \left\lfloor\frac{n+r}{2}\right\rfloor = n - \xi(n-r)$. This completes the proof.

Theorem 2.8. *(Cassini's Identity) Let n be nonnegative integer. Then, we have*

$$\left(\frac{a}{b}\right)^{\xi(n-1)}\mathbb{F}_{n-1}(x)\mathbb{F}_{n+1}(x) - \left(\frac{a}{b}\right)^{\xi(n)}\mathbb{F}_n{}^2(x) = -(-g(x))^{n-1}\frac{a}{b}$$

Proof. The proof can be seen in an obvious way by taking $r = 1$ in the Catalan's identity.

Theorem 2.9. *(d'Ocagne's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have*

$$a^{\xi(nr+n)}b^{\xi(nr+r)}\mathbb{F}_n(x)\,\mathbb{F}_{r+1}(x) - a^{\xi(nr+r)}b^{\xi(nr+n)}\mathbb{F}_{n+1}(x)\,\mathbb{F}_r(x) = (-g(x))^r\, a^{\xi(n-r)}\mathbb{F}_{n-r}(x)$$

where $\mathbb{F}_n(x)$ is the nth bi-periodic generalized Fibonacci polynomial.

Proof. Using Binet formula for nth bi-periodic generalized Fibonacci polynomial given in Theorem 2.4, we have

$$a^{\xi(nr+n)}b^{\xi(nr+r)}\mathbb{F}_n(x)\,\mathbb{F}_{r+1}(x) - a^{\xi(nr+r)}b^{\xi(nr+n)}\mathbb{F}_{n+1}(x)\,\mathbb{F}_r(x)$$

$$\begin{aligned}
&= a^{\xi(nr+n)}b^{\xi(nr+r)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r+1)}}{(ab)^{\lfloor\frac{r+1}{2}\rfloor}}\right)\left(\frac{\gamma^n(x)-\delta^n(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{r+1}(x)-\delta^{r+1}(x)}{\gamma(x)-\delta(x)}\right)\\
&- a^{\xi(nr+r)}b^{\xi(nr+n)}\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\right)\left(\frac{\gamma^{n+1}(x)-\delta^{n+1}(x)}{\gamma(x)-\delta(x)}\right)\left(\frac{\gamma^{r}(x)-\delta^{r}(x)}{\gamma(x)-\delta(x)}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{1-\xi(n)-\xi(r+1)+\xi(nr+n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{r+1}{2}\rfloor}}\left(\frac{\gamma^{n+r+1}(x)-\gamma^n(x)\,\delta^{r+1}(x)-\delta^n(x)\,\gamma^{r+1}(x)+\delta^{n+r+1}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{1-\xi(n+1)-\xi(r)+\xi(nr+r)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{r}{2}\rfloor}}\left(\frac{\gamma^{n+r+1}(x)-\gamma^{n+1}(x)\,\delta^{r}(x)-\delta^{n+1}(x)\,\gamma^{r}(x)+\delta^{n+r+1}(x)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{\xi(n-r)-\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma(x)\,\delta^{n-r}(x)+\delta(x)\,\gamma^{n-r}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{\xi(n-r)-\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{ab^{\xi(nr+r)}a^{\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma(x)\,\delta^{n-r}(x)+\delta(x)\,\gamma^{n-r}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&- \frac{ab^{\xi(nr+n)}a^{\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}\left(\frac{\gamma^{n+r+1}(x)+\delta^{n+r+1}(x)-(\gamma(x)\,\delta(x))^r\left(\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a(ab)^{-r}}{(ab)^{\frac{n-r-\xi(n-r)}{2}}}\left(\frac{(\gamma(x)\,\delta(x))^r\left(-\gamma(x)\,\delta^{n-r}(x)-\delta(x)\,\gamma^{n-r}(x)+\gamma^{n-r+1}(x)+\delta^{n-r+1}(x)\right)}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a(ab)^{-r}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{(-g(x)\,ab)^r(\gamma(x)-\delta(x))(\gamma^{n-r}(x)-\delta^{n-r}(x))}{(\gamma(x)-\delta(x))^2}\right)\\
&= \frac{a(-g(x))^r}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{\gamma^{n-r}(x)-\delta^{n-r}(x)}{\gamma(x)-\delta(x)}\right)\\
&= (-g(x))^r\, a^{\xi(n-r)}\mathbb{F}_{n-r}(x)
\end{aligned}$$

where

$$\xi(n)+\xi(r+1)-2\xi(nr+n) = \xi(n+1)+\xi(r)-2\xi(nr+r) = 1-\xi(n-r)$$

$$\xi(n-r) = \xi(nr+n)+\xi(nr+r)$$

$$\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r=\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{r+1}{2}\right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r=\left\lfloor\frac{n+1}{2}\right\rfloor+\left\lfloor\frac{r}{2}\right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2}=\left\lfloor\frac{n-r}{2}\right\rfloor.$$

This completes the proof.

3. Conclusion and Suggestion

The most interesting applications of the Fibonacci numbers have been on its generalizations, also called families of Fibonacci numbers. Large classes of polynomials are emerged as the well-known generalization of Fibonacci numbers. In this paper, the bi-periodic generalized Fibonacci polynomials, which generalize well-known Fibonacci polynomials, the $h(x)$-Fibonacci polynomials, the Fibonacci polynomials with two variable, the Pell polynomials, the Jacobsthal polynomials, the Fermat polynomials, the Chebyshev second kind polynomials, the Morgan-Voyce first kind polynomials, the Vieta polynomials, are defined. Also the bi-periodic Fibonacci polynomials, the bi-periodic $h(x)$-Fibonacci polynomials, the bi-periodic Fibonacci polynomials with two variable, the bi-periodic Pell polynomials, the bi-periodic Jacobsthal polynomials, the bi-periodic Fermat polynomials, the bi-periodic Chebyshev second kind polynomials, the bi-periodic Morgan-Voyce first kind polynomials, the bi-periodic Vieta polynomials are presented. Binet formulas that allow us to calculate the nth term of these polynomial sequences and some properties of their consecutive terms are given. Also generating functions, Catalan's identity, Cassini's identity, and d'Ocagne's identity are obtained.

It would be interesting to study these polynomials in matrix theory. More general formulas that allow us to calculate the nth terms of these polynomial sequences and sums formulas can be explored.

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