TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 185-201
ISSN: 2587–0971

http:/dergipark.gov.tr/tjos

New Variants of Hermite-Hadamard Type Inequalities via Generalized Fractional Operator for Differentiable Functions

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Abstract. The main motivation of this study is to present new Hermite-Hadamard (HH) type inequalities via a certain fractional operators. We establish two new identities and give new estimations of HH- type inequalities for differentiable and convex mapping via Katugampola-fractional operators. Here, we gave new Lemmas having identities for differentiable functions and construct related inequalities. Main findings of this study would provide elegant connections and general variants of well known results established recently. These results can be extended to different kinds of convex functions as well as pre-invex functions.

1. Introduction

Convexity is a very functional concept in programming, statistics and numerical analysis as in many different branches of mathematics. In theory of inequality, the concept of convexity exists in the proof of many classical inequalities, but has been a source of inspiration for many new and useful inequalities.

Definition 1.1. *[22]. The function $f : [c_1, c_2] \to \Re$, is said to be convex, if we have*

$$f\left(t\kappa + (1-t)\tau\right) \leq t f(\kappa) + (1-t)\, f(\tau)$$

for all $\kappa, \tau \in [c_1, c_2]$ and $t \in [0,1]$.

In addition to the use of convex functions in many fields, inequality has increased its reputation in theory with the Hermite-Hadamard inequality (See [22]). This celebrated inequality can be stated as: If a mapping $\Upsilon : J \subseteq \Re \to \Re$ is a convex function on J and $r, s \in J$, $r < s$, then

$$\Upsilon\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Upsilon(\lambda)d\lambda \leq \frac{\Upsilon(r)+\Upsilon(s)}{2}.$$

Fractional calculus is a good expansion of the concept of derivative operator from integer order n to arbitrary order a. Fractional derivative operators are accepted as the inverse of fractional integral operators. Recently, the multiplicity of applications in many fields of engineering, physics, statistics and mathematics

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Received: 24 November 2022; Accepted: 15 December 2022; Published: 30 December 2022
Keywords. Convex functions; Hermite-Hadamard inequalities; Katugampola-fractional integrals.
2010 *Mathematics Subject Classification*. 26A33, 26A51, 26D10, 26D07.
Cited this article as: Nasir J. Butt SI. Dokuyucu MA. and Akdemir AO. New Variants of Hermite-Hadamard type inequalities Via Generalized fractional operator for Differentiable Functions, Turkish Journal of Science, 2022, 7(3), 185-201.

has led to the study of fractional integrals by many researchers. The fact that they are a more effective tool than the results in classical analysis has resulted in more use of these operators on real world problems. Since the definition of the convex functions has been given as an inequality, this concept has established a powerful link between convexity and inequalities. It is now become a trending aspect of mathematical research to generalize classical known results via fractional integral operator. Although fractional analysis is basically a generalization of classical analysis, it has developed rapidly with the concepts of fractional order operators. Fractional analysis has recently become a popular topic with its applications in many fields such as modeling, physics, approximation theory, engineering, control theory and mathematical biology, based on applied mathematics problems (see [1], [3], [8], [9]-[11], [17], [18]-[21] and [23]-[26]).

Recently in [14], the author introduced a new concept to unify Riemann-Liouville and Hadamard fractional integral operators which a certain general form for fractional integral operators. Also the conditions are given so that the operator is bounded in an extended Lebesgue measurable space. The corresponding fractional derivative approach to this new generalized operator can be seen in [15]. Moreover, Katugampola worked for the Mellin transforms of the fractional integrals and derivatives (see [16]).

Definition 1.2. *([14]) Let $[\kappa, \tau] \subset \Re$ be a finite interval. Then the left-sided and right-sided Katugampola fractional integrals of order $\xi > 0$ of $\Upsilon \in X_c^\nu(\kappa^\nu, \tau^\nu)$ are defined as follows:*

$$({}^\nu I_{\kappa+}^\xi \Upsilon)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)} \int_\kappa^x \frac{\Upsilon(\lambda)}{(x-\lambda)^{1-\xi}} \lambda^{\nu-1} d\lambda, \ x > \kappa$$

and

$$({}^\nu I_{\tau-}^\xi \Upsilon)(x) = \frac{\nu^{1-\xi}}{\Gamma(\xi)} \int_x^\tau \frac{\Upsilon(\lambda)}{(\lambda-x)^{1-\xi}} \lambda^{\nu-1} d\lambda, \ x < \tau,$$

with $\kappa < x < \tau$ and $\nu > 0$, if the integrals exist.

Theorem 1.3. *([14]) If $\xi > 0$ and $\nu > 0$, then for $x > \kappa$*

$$1) \lim \nu \to 1 ({}^\nu I_{\kappa^+}^\xi \Upsilon)(x) = (J_{\kappa^+}^\xi \Upsilon)(x)$$

$$2) \lim \nu \to (0^+) ({}^\nu I_{\kappa^+}^\xi \Upsilon)(x) = (H_{\kappa^+}^\xi \Upsilon)(x).$$

The main motivation point of the study is to prove the HH type inequalities with specific and general forms for the functions whose absolute values of derivatives are convex and concave functions with the help of the fractional integral operator, which has a general kernel structure. The main results are reduced to the results available in the literature in some special cases, as well as giving new approximations and estimates for differentiable and convex functions. To obtain our results, we used some known proof methods alongside classical inequalities such as the Hölder inequality, Power mean inequality, and Weighted Hölder inequality.

2. Hermite-Hadamard Type inequalities for Katugampola-Fractional Integrals

We will start with the following identities that will be useful to prove our main findings via Katugampola fractional integrals:

Lemma 2.1. *Let $\xi \in (0,1)$ and $\nu > 0$ and $f : [\kappa^\nu, \tau^\nu] \to \Re$ be a twice differentiable mapping on (κ^ν, τ^ν) with $0 < \kappa^\nu < \tau^\nu$. Then the following equality holds for Katugampola fractional integral operators:*

$$
\begin{aligned}
|A| &= \frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu - \kappa^\nu\right)^\xi}\left(\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f\left(\tau^\nu\right) + \left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f\left(\kappa^\nu\right)\right) - f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right) \\
&= \frac{(\tau^\nu - \kappa^\nu)}{4}\Big[\int_0^1 t^{\nu\xi} t^{\nu-1} f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt + \int_0^1 t^{\nu\xi} t^{\nu-1} f'\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big] \\
&= \frac{(\tau^\nu - \kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt \\
&+ \int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big].
\end{aligned}
$$

Proof. By applying integration by parts to the right hand side of the equality, we have

$$
\begin{aligned}
& k_1 \\
=\ & \int_0^1 t^{\nu\xi} t^{\nu-1} f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt \\
=\ & f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)\frac{t^{\nu(\xi+1)}}{\nu(\xi+1)}\Big|_0^1 \\
& -\int_0^1 \frac{t^{\nu(\xi+1)}}{\nu(\xi+1)} f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)\nu t^{\nu-1}\Big(\frac{\kappa^\nu - \tau^\nu}{2}\Big)dt \\
=\ & \frac{1}{\nu(\xi+1)} f'\Big(\frac{\kappa^\nu + \tau^\nu}{2}\Big) \\
& -\frac{\left(\frac{\kappa^\nu - \tau^\nu}{2}\right)}{(\xi+1)}\int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt.
\end{aligned}
$$

Similarly, we can write

$$
\begin{aligned}
k_2 &= \int_0^1 t^{\nu\xi} t^{\nu-1} f'\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt \\
&= \frac{1}{\nu(\xi+1)} f'\Big(\frac{\kappa^\nu + \tau^\nu}{2}\Big) \\
&\quad -\frac{\left(\frac{\tau^\nu - \kappa^\nu}{2}\right)}{(\xi+1)}\int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt.
\end{aligned}
$$

Now, by taking into account $(k_1 - k_2)$, we obtain

$$
\begin{aligned}
& (k_1 - k_2) \\
=\ & \frac{(\tau^\nu - \kappa^\nu)}{2(\xi+1)}\Big[\int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt \\
& + \int_0^1 t^{\nu(\xi+1)} t^{\nu-1} f''\Big(\frac{t^\nu}{2}\tau^\nu + \frac{2-t^\nu}{2}\kappa^\nu\Big)dt\Big].
\end{aligned}
\tag{1}
$$

On the other hand, we have

$$
I_1 = \int_0^1 (t^\nu)^\xi t^{\nu-1} f'\Big(\frac{t^\nu}{2}\kappa^\nu + \frac{2-t^\nu}{2}\tau^\nu\Big)dt
$$

$$= (t^\nu)^\xi t^{\nu-1}\frac{f\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)}{\nu t^{\nu-1}\Big(\frac{\kappa^\nu-\tau^\nu}{2}\Big)}\Big|_0^1-\int_0^1\xi(t^\nu)^{\xi-1}\nu t^{\nu-1}\frac{t^{\nu-1}f\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)}{\nu t^{\nu-1}\Big(\frac{\kappa^\nu-\tau^\nu}{2}\Big)}dt$$
$$= \frac{-2f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)}{\nu(\kappa^\nu-\tau^\nu)}+\frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big(\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)_+^{\frac{1}{\nu}}}\Big)f\big(\tau^\nu\big)\Big).$$

By a similar way, it is easy to see that

$$\begin{aligned} I_2 &= \int_0^1(t^\nu)^\xi t^{\nu-1}f'\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)dt \\ &= \frac{2f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)}{\nu(\tau^\nu-\kappa^\nu)} \\ &\quad -\frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big(\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)_-^{\frac{1}{\nu}}}\Big)f\big(\kappa^\nu\big)\Big). \end{aligned}$$

Thus,

$$\begin{aligned} &(I_1-I_2) \\ &= \frac{-4f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)}{\nu(\tau^\nu-\kappa^\nu)}+\frac{2^{\xi+1}\Gamma(\xi+1)}{\nu^{1-\xi}(\tau^\nu-\kappa^\nu)^{\xi+1}} \\ &\quad\times\Big[\Big(\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)_+^{\frac{1}{\nu}}}\Big)f\big(\tau\big)\Big)+{}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^-}f(\kappa^\nu)\Big]. \end{aligned} \tag{2}$$

Multiplying (1) and (2) by $\frac{\nu(\tau^\nu-\kappa^\nu)}{4}$, we get the desired result. □

Lemma 2.2. *Let $\xi\in(0,1)$ and $\nu>0$ and $f:[\kappa^\nu,\tau^\nu]\to\Re$ be a twice differentiable mapping on (κ^ν,τ^ν) with $0<\kappa<\tau$. Then, the following equality holds for Katugampola fractional integral operators:*

$$|B| = \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)dt$$
$$+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}f''\Big(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\Big)dt\Big]$$
$$=\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)_+^{\frac{1}{\nu}}}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)_-^{\frac{1}{\nu}}}\Big)f\big(\kappa^\nu\big)-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)\Big].$$

Proof. By integration by parts for the right hand side of the equality, we have

$$\begin{aligned} I_1 &= \int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)}+\frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\Big[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu}+\frac{2\xi}{\tau^\nu-\kappa^\nu}\int_0^1(1-t^\nu)^{\xi-1}t^{\nu-1}f(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu)\Big]. \end{aligned}$$

By changing of the variables, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\kappa^\nu + \frac{1+t^\nu}{2}\tau^\nu\Big)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad \times\Big[\frac{-2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi}{(\tau^\nu-\kappa^\nu)^{\xi+1}} \int_{(\frac{\kappa^\nu+\tau^\nu}{2})^{\frac{1}{\nu}}}^{\tau} (\tau^\nu-u^\nu)^{\xi-1} u^{\nu-1} f(u^\nu)du\Big] \end{aligned}$$

Multiplying the resulting equality by $\frac{\Gamma(\xi)\nu^{1-\xi}}{\Gamma(\xi)\nu^{1-\xi}}$, we obtain

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\kappa^\nu + \frac{1+t^\nu}{2}\tau^\nu\Big)dt \\ &= \frac{-2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\Big[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big)\Big]. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 (1-t^\nu)^{\xi+1} t^{\nu-1} f''\Big(\frac{1-t^\nu}{2}\tau^\nu + \frac{1+t^\nu}{2}\kappa^\nu\Big)dt \\ &= \frac{2f'(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)} + \frac{2(\xi+1)}{\tau^\nu-\kappa^\nu} \\ &\quad -\Big[\frac{2f(\frac{\kappa^\nu+\tau^\nu}{2})}{\tau^\nu-\kappa^\nu} + \frac{2^{\xi+1}\xi\Gamma(\xi)}{(\tau^\nu-\kappa^\nu)^{\xi+1}\nu^{1-\xi}}\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big]. \end{aligned}$$

Namely,

$$\begin{aligned} & I_1 + I_2 \\ = & -\frac{8(\xi+1)f(\frac{\kappa^\nu+\tau^\nu}{2})}{\nu(\tau^\nu-\kappa^\nu)^2} + \frac{2^{\xi+2}\Gamma(\xi+2)}{(\upsilon^\nu-\kappa^\nu)^{\xi+2}\nu^{1-\xi}} \\ & \times\Big[\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big) + \Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big] \end{aligned}$$

Now, multiplying both sides by $\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}$, we provide

$$\begin{aligned} & \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\big(I_1+I_2\big) \\ = & -f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big) \\ & +\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi}}{(\tau^\nu-\kappa^\nu)^{\xi}}\Big[\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^\nu\big) + \Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^\nu\big)\Big]. \end{aligned}$$

Which completes the proof. □

Theorem 2.3. *Suppose that $f : [\kappa^\nu, \tau^\nu] \to \mathfrak{R}$ be a twice differentiable function on (κ^ν, τ^ν) with $0 \leq \kappa < \tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^\xi}\left(\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f\left(\tau^\nu\right)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f\left(\kappa^\nu\right)\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu(\xi+2)}\right)\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Proof. By using right hand side of the Lemma (2.1), we can write

$$\begin{aligned}|A| \quad &\leq \quad \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\int_0^1 t^{\xi\nu+\nu}t^{\nu-1}\left[\frac{t^\nu}{2}\left|f''(\kappa^\nu)\right|+\frac{2-t^\nu}{2}\left|f''(\tau^\nu)\right|\right]dt\right.\\ &\left.+\int_0^1 t^{\xi\nu+\nu}\left[\frac{t^\nu}{2}\left|f''(\tau^\nu)\right|+\frac{2-t^\nu}{2}\left|f''(\kappa^\nu)\right|\right]dt\right].\end{aligned}$$

By making use of the necessary calculations, we get

$$|A| \leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu(\xi+2)}\right)\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Which completes the proof. □

Theorem 2.4. *Suppose that $f : [\kappa^\nu, \tau^\nu] \to \mathfrak{R}$ be a twice differentiable function on (κ^ν, τ^ν) with $0 \leq \kappa < \tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^\xi}\left(\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f\left(\tau^\nu\right)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f\left(\kappa^\nu\right)\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left(\frac{1}{\nu s(\xi+2)-s+1}\right)^{\frac{1}{q}}$$

$$\left[\left(\frac{1}{2(\nu+1)}|f''(\kappa^\nu)|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\tau^\nu)|^r\right)^{\frac{1}{p}}+\left(\frac{1}{2(\nu+1)}|f''(\tau^\nu)|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\kappa^\nu)|^r\right)^{\frac{1}{p}}\right].$$

for $p > 1$ and $q > 1$.

Proof. From the right hand side of Lemma (2.1), we have

$$\begin{aligned}&|A|\\ \leq \quad &\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}\left|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)+f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\right|dt\right]\end{aligned}$$

By using the Hölder inequality, we get

$$\begin{aligned}|A| \leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}&\left[\Big(\int_0^1 (t^{\nu(\xi+2)-1})^q dt\Big)^{\frac{1}{q}}\Big(\int_0^1\left|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\right|^p dt\Big)^{\frac{1}{p}}\right.\\ &\left.+\Big(\int_0^1 \left(t^{\nu(\xi+2)-1}\right)^q dt\Big)^{\frac{1}{q}}\Big(\int_0^1\left|f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\right|^p dt\Big)^{\frac{1}{p}}\right]\end{aligned}$$

$$\leq \frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Big(\int_0^1 (t^{\nu(\xi+2)-1})^q dt\Big)^{\frac{1}{q}}$$

$$\Big[\int_0^1 \Big(\frac{t^{\nu}}{2}|f''(\kappa^{\nu})|^p+\frac{2-t^{\nu}}{2}|f''(\tau^{\nu})|^p\Big)dt+\int_0^1 \Big(\frac{t^{\nu}}{2}|f''(\tau^{\nu})|^p+\frac{2-t^{\nu}}{2}|f''(\kappa^{\nu})|^p\Big)dt\Big]^{\frac{1}{p}}$$

Thus, we provide

$$|A|\leq \frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Big(\frac{1}{\nu q(\xi+2)-q+1}\Big)^{\frac{1}{q}}$$

$$\Bigg[\Big(\frac{1}{2(\nu+1)}|f''(\kappa^{\nu})|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\tau^{\nu})|^r\Big)^{\frac{1}{r}}+\Big(\frac{1}{2(\nu+1)}|f''(\tau^{\nu})|^r+\frac{2\nu+1}{2(\nu+1)}|f''(\kappa^{\nu})|^r\Big)^{\frac{1}{r}}\Bigg].$$

This completes the proof. □

Theorem 2.5. *If $f:[\kappa^{\nu},\tau^{\nu}]\to\Re$ be differentiable function on $(\kappa^{\nu},\tau^{\nu})$ with $\kappa^{\nu}<\tau^{\nu}$ and $f''\in L_1[\kappa^{\nu},\tau^{\nu}]$. If $|f''|$ is a concave function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\big(\tau^{\nu}-\kappa^{\nu}\big)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)$$

$$\leq \frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)\Bigg[\Bigg|f''\Bigg(\frac{\kappa^{\nu}}{2\nu(\xi+3)}+\Bigg(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\Bigg)\frac{\tau^{\nu}}{2}\Bigg)\Bigg|$$

$$+\Bigg|f''\Bigg(\frac{\tau^{\nu}}{2\nu(\xi+3)}+\Bigg(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\Bigg)\frac{\kappa^{\nu}}{2}\Bigg)\Bigg|\Bigg]\Bigg].$$

Proof. From Lemma 2.1, we have

$$\Bigg|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\big(\tau^{\nu}-\kappa^{\nu}\big)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(b^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)\Bigg|$$

$$\leq \frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)\Big|dt$$

$$+\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\Big|dt\Bigg]$$

By applying Jensen Integral inequality, we get

$$|A|\leq \frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\int_0^1 t^{\nu(\xi+2)-1}dt\Big)\Bigg|f''\Bigg(\frac{\int_0^1 t^{\nu(\xi+2)-1}\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)dt}{\int_0^1 t^{\nu(\xi+2)-1}dt}\Bigg)\Bigg|$$

$$+\Big(\int_0^1 t^{\nu(\xi+2)-1}dt\Big)\Bigg|f''\Bigg(\frac{\int_0^1 t^{\nu(\xi+2)-1}\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)dt}{\int_0^1 t^{\nu(\xi+2)-1}dt}\Bigg)\Bigg|\Bigg]$$

$$=\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)\Bigg[\Bigg|f''\Bigg(\frac{\kappa^{\nu}}{2\nu(\xi+3)}+\Bigg(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\Bigg)\frac{\tau^{\nu}}{2}\Bigg)\Bigg|$$

$$+\Bigg|f''\Bigg(\frac{\tau^{\nu}}{2\nu(\xi+3)}+\Bigg(\frac{2}{\nu(\xi+2)}-\frac{1}{\nu(\xi+3)}\Bigg)\frac{\kappa^{\nu}}{2}\Bigg)\Bigg|\Bigg]\Bigg].$$

Which completes the proof. □

Theorem 2.6. *If $f:[\kappa^{\nu},\tau^{\nu}]\to\Re$ be differentiable function on $(\kappa^{\nu},\tau^{\nu})$ with $\kappa^{\nu}<\tau^{\nu}$ and $f''\in L_1[\kappa^{\nu},\tau^{\nu}]$. If $|f''|^q$ is a convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^{\nu}-\kappa^{\nu}\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)$$

$$\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^{\nu})|^q+|f''(\tau^{\nu})|^q}{\nu(\xi+2)}\Bigg]^{\frac{1}{q}}\Bigg].$$

Proof. From Lemma 2.1, we have

$$\left|\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^{\nu}-\kappa^{\nu}\right)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(\tau^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)\right|$$

$$\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)\Big|dt$$

$$+\int_0^1 t^{\nu(\xi+1)}t^{\nu-1}\Big|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\Big|dt\Bigg]$$

By applying Power-mean inequality, we get

$$|A|\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^{\nu}}{2}\kappa^{\nu}+\frac{2-t^{\nu}}{2}\tau^{\nu}\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Bigg]$$

By using convexity of $|f''|^q$, we get

$$|A|\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}$$

$$\Bigg[\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\frac{t^{\nu}}{2}|f''(\kappa^{\nu})|^q dt+\int_0^1\frac{2-t^{\nu}}{2}|f''(\tau^{\nu})|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 t^{\nu(\xi+2)-1}\Big)^{1-\frac{1}{q}}\Big(\int_0^1 t^{\nu(\xi+2)-1}\frac{t^{\nu}}{2}|f''(\tau^{\nu})|^q dt+\int_0^1\frac{2-t^{\nu}}{2}|f''(\kappa^{\nu})|^q dt\Big)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^{\nu})|^q}{2}\Big(\frac{2}{\nu\xi+2\nu}\Big)+\frac{|f''(\tau^{\nu})|^q}{2}\Big(\frac{2}{\nu\xi+2\nu}\Big)\Bigg]^{\frac{1}{q}}\Bigg].$$

By simplfying the above inequality, we obtain

$$|A\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^{\nu})|^q+|f''(\tau^{\nu})|^q}{\nu(\xi+2)}\Bigg]^{\frac{1}{q}}\Bigg].$$

Which completes the proof. □

Theorem 2.7. *Let $f : I^\circ \subset \Re \to \Re$ be a differentiable mapping on I° and $\kappa^\nu, \tau^\nu \in I^\circ$ with $\kappa^\nu < \tau^\nu$ and $q \geq 1$. If the mapping $|f''|^q$ is convex on the interval (κ^ν, τ^ν), then the following inequality holds for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\left(\tau^\nu-\kappa^\nu\right)^\xi}\left(\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\right)f\left(\tau^\nu\right)+\left({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\right)f\left(\kappa^\nu\right)\right)-f\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\left(\frac{1}{(-p+2p\nu+p\xi\nu+1)(2p\nu+p\xi\nu-p+2)}\right)^{\frac{1}{p}}\right.$$

$$\left[\left(\frac{|f''(\kappa^\nu)|^q}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^\nu)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|f''(\tau^\nu)|^q}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^\nu)|^q\right)^{\frac{1}{q}}\right]+\left(\frac{1}{2p\nu+p\nu\xi-p+2}\right)^{\frac{1}{p}}$$

$$\left.\left[\left(\frac{|f''(\kappa^\nu)|^q}{2(\nu+2)}+\frac{(\nu+1)|f''(\tau^\nu)|^q}{2(\nu+2)}\right)^{\frac{1}{q}}+\left(\frac{|f''(\tau^\nu)|^q}{2(\nu+2)}+\frac{(\nu+1)|f''(\kappa^\nu)|^q}{2(\nu+2)}\right)^{\frac{1}{q}}\right]\right].$$

Proof. From Lemma 2.1, we can write

$$|A| \leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\left[\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)dt\right.$$

$$\left.+\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\left(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\right)dt\right].$$

Let us denote

$$k_1=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)dt.$$

By using Hölder-Işcan inequality, we have

$$|A| \leq \left(\int_0^1(1-t)|t^{\nu(\xi+2)-1}|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t|t^{\nu(\xi+2)-1}|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1(1-t)(t^{p\nu(\xi+2)-p})dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\frac{t^\nu}{2}\left|f''(\kappa^\nu)\right|^q dt+\int_0^1(1-t)\frac{2-t^\nu}{2}\left|f''(\tau^\nu)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t(t^{p\nu(\xi+2)-p})dt\right)^{\frac{1}{p}}\left(\int_0^1 t\frac{t^\nu}{2}\left|f''(\kappa^\nu)\right|^q dt+\int_0^1 t\frac{2-t^\nu}{2}\left|f''(\tau^\nu)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{(-p+2p\nu+p\xi\nu+1)(-p+2p\nu+p\xi\nu+2)}\right)^{\frac{1}{p}}$$

$$\left(\frac{1}{2(\nu+1)(\nu+2)}|f''(\kappa^\nu)|^q+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^\nu)|^q\right)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{2p\nu+p\xi\nu+2}\Big)^{\frac{1}{p}}\Big(\frac{1}{2(\nu+2)}|f''(\kappa^{\nu})|^{q}+\frac{\nu+1}{2(\nu+2)}|f''(\tau^{\nu})|^{q}\Big)^{\frac{1}{q}}.$$

Similarly,

$$k_2=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu}\Big)dt.$$

By using Hölder-Işcan inequality, we have

$$|A|\leq\Big(\int_0^1(1-t)|t^{\nu(\xi+2)-1}|^{p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\Big|f''(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 t|t^{\nu(\xi+2)-1}|^{p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\Big|f''(\frac{t^{\nu}}{2}\tau^{\nu}+\frac{2-t^{\nu}}{2}\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$
$$\leq\Big(\int_0^1(1-t)(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1(1-t)\frac{t^{\nu}}{2}\Big|f''(\tau^{\nu})\Big|^{q}dt+\int_0^1(1-t)\frac{2-t^{\nu}}{2}\Big|f''(\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 t(t^{p\nu(\xi+2)-p})dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t\frac{t^{\nu}}{2}\Big|f''(\tau^{\nu})\Big|^{q}dt+\int_0^1 t\frac{2-t^{\nu}}{2}\Big|f''(\kappa^{\nu})\Big|^{q}dt\Big)^{\frac{1}{q}}$$
$$\leq\Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(-p+2p\nu+p\xi\nu+2)}\Big)^{\frac{1}{p}}$$
$$\Big(\frac{1}{2(\nu+1)(\nu+2)}|f''(\tau^{\nu})|^{q}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{1}{2p\nu+p\xi\nu+2}\Big)^{\frac{1}{p}}\Big(\frac{1}{2(\nu+2)}|f''(\tau^{\nu})|^{q}+\frac{\nu+1}{2(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}.$$

Now, k_1+k_2

$$|A|\leq\frac{(\tau^{\nu}-\kappa^{\nu})^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(-p+2p\nu+p\xi\nu+1)(2p\nu+p\xi\nu-p+2)}\Big)^{\frac{1}{p}}$$
$$\Bigg[\Big(\frac{|f''(\kappa^{\nu})|^{q}}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\tau^{\nu})|^{q}\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{|f''(\tau^{\nu})|^{q}}{2(\nu+1)(\nu+2)}+\frac{\nu^2+3\nu+1}{2(\nu+1)(\nu+2)}|f''(\kappa^{\nu})|^{q}\Big)^{\frac{1}{q}}\Bigg]$$
$$+\Big(\frac{1}{2p\nu+p\nu\xi-p+2}\Big)^{\frac{1}{p}}$$
$$\Bigg[\Big(\frac{|f''(\kappa^{\nu})|^{q}}{2(\nu+2)}+\frac{(\nu+1)|f''(\tau^{\nu})|^{q}}{2(\nu+2)}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\tau^{\nu})|^{q}}{2(\nu+2)}+\frac{(\nu+1)|f''(\kappa^{\nu})|^{q}}{2(\nu+2)}\Big)^{\frac{1}{q}}\Bigg]\Bigg].$$

Which is the desired result. □

Theorem 2.8. *Let $f:I^{\circ}\subset\Re\to\Re$ be a differentiable mapping on I° and $\kappa^{\nu},\tau^{\nu}\in I^{\circ}$ with $\tau^{\nu}<\kappa^{\nu}$ and $q\geq 1$. If the mapping $|f''|^{q}$ is convex on the interval $(\kappa^{\nu},\tau^{\nu})$, then the following inequality holds for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^{\xi-1}}{\big(\tau^{\nu}-\kappa^{\nu}\big)^{\xi}}\Big(\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{+}}\Big)f\big(b^{\nu}\big)+\Big({}^{\nu}I^{\xi}_{\left(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\right)^{\frac{1}{\nu}}_{-}}\Big)f\big(\kappa^{\nu}\big)\Big)-f\Big(\frac{\kappa^{\nu}+\tau^{\nu}}{2}\Big)$$

$$\leq \frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\kappa^\nu)|^q\Big)^{\frac{1}{q}}\Bigg]$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\frac{|f''(\tau^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^\nu)|^q\Bigg)^{\frac{1}{q}}\Bigg]\Bigg].$$

Proof. Let us denote

$$k_1=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)dt$$

By using Improved Power mean inequality, we get

$$|A|\leq\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 tt^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 tt^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\kappa^\nu+\frac{2-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\leq\Big(\frac{1}{2\nu+\xi\nu}-\frac{1}{2\nu+\xi\nu+1}\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^\nu}{2}|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{2-t^\nu}{2}|f''(b^\nu)|^q dt\Bigg)^{\frac{1}{q}}+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\int_0^1 t^{\nu(\xi+2)}\frac{t^\nu}{2}|f''(\kappa^\nu)|^q dt+\int_0^1 t^{\nu(\xi+2)}\frac{2-t^\nu}{2}|f''(\tau^\nu)|^q dt\Bigg)^{\frac{1}{q}}.$$

By taking into account the facts that

$$\int_0^1(1-t)t^{\nu(\xi+2)-1}(1-\frac{t^\nu}{2})dt=\int_0^1(1-t)t^{\nu(\xi+2)-1}dt-\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^\nu}{2}dt$$

$$=\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}.$$

It is clear to see that

$$|A|\leq\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\frac{|f''(\kappa^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu)+1}\Big)|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Big(\frac{|f''(\kappa^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^\nu)|^q\Big)^{\frac{1}{q}}.$$

Let

$$k_2=\int_0^1 t^{\nu\xi+\nu}t^{\nu-1}f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)dt$$

By using Improved Power mean inequality

$$|A|\le\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)t^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 tt^{\nu(\xi+2)-1}dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1 tt^{\nu(\xi+2)-1}\Big|f''\Big(\frac{t^\nu}{2}\tau^\nu+\frac{2-t^\nu}{2}\kappa^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$\le\Big(\frac{1}{2\nu+\xi\nu}-\frac{1}{2\nu+\xi\nu+1}\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{t^\nu}{2}|f''(\tau^\nu)|^q dt$$

$$+\int_0^1(1-t)t^{\nu(\xi+2)-1}\frac{2-t^\nu}{2}|f''(\kappa^\nu)|^q dt\Bigg)^{\frac{1}{q}}+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\int_0^1 t^{\nu(\xi+2)}\frac{t^\nu}{2}|f''(\tau^\nu)|^q dt+\int_0^1 t^{\nu(\xi+2)}\frac{2-t^\nu}{2}|f''(\kappa^\nu)|^q dt\Bigg)^{\frac{1}{q}}.$$

By computing the above integrals, we have

$$|A|\le\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}$$

$$\Bigg(\frac{|f''(\tau^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu)+1}\Big)|f''(\kappa^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Big(\frac{|f''(\tau^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^\nu)|^q\Big)^{\frac{1}{q}}.$$

Now, k_1+k_2

$$|A|\le\frac{(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Big(\frac{|f''(\tau^\nu)|^q}{2\nu(\xi+3)(\xi\nu+3\nu+1)}$$

$$+\Big(\frac{1}{(2\nu+\xi\nu)(2\nu+\xi\nu+1)}-\frac{1}{2\nu(\xi+3)(\xi\nu+3\nu+1)}\Big)|f''(\kappa^\nu)|^q\Big)^{\frac{1}{q}}\Bigg]$$

$$+\Big(\frac{1}{\nu(\xi+2)+1}\Big)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{|f''(\kappa^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\tau^\nu)|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\frac{|f''(\tau^\nu)|^q}{2(3\nu+\xi\nu+1)}+\frac{\xi\nu+4\nu+1}{2(2\nu+\xi\nu+1)(3\nu+\xi\nu+1)}|f''(\kappa^\nu)|^q\Bigg)^{\frac{1}{q}}\Bigg]\Bigg].$$

Which is the desired result. □

Theorem 2.9. *Suppose that $f : [\kappa^\nu, \tau^\nu] \to \mathfrak{R}$ be a twice differentiable function on (κ^ν, τ^ν) with $0 \leq \kappa < \tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

Proof. By using the property of modulus on R.H.S of lemma (2.2), we can write

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu$$

$$+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\Big)\Big|dt\Big]$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[\frac{1-t^\nu}{2}|f''(\kappa^\nu)|+\frac{1+t^\nu}{2}|f''(\tau^\nu)|\Big]dt$$

$$+\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[\frac{1-t^\nu}{2}|f''(\tau^\nu)|+\frac{1+t^\nu}{2}|f''(\kappa^\nu)|\Big]dt\Big]$$

$$=\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1(1-t^\nu)^{\xi+1}t^{\nu-1}\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big]dt\Big]$$

$$=\frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big]$$

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big]\Big[|f''(\kappa^\nu)|+|f''(\tau^\nu)|\Big].$$

This completes the proof. □

Theorem 2.10. *Suppose that $f : [\kappa^\nu, \tau^\nu] \to \mathfrak{R}$ be a twice differentiable function on (κ^ν, τ^ν) with $0 \leq \kappa < \tau$. If $|f''|$ is convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{s(\xi+1)+1}\Big)^{\frac{1}{s}}\Big[\Big(\frac{|f''(\kappa^\nu)|^q+3|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}+\Big(\frac{3|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}\Big]\Big]$$

for $r > 1$ and $s > 1$.

Proof. Using Hölder Inequality in lemma (2.2), we get

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1|(1-t^\nu)^{\xi+1}|^p t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}a^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 |(1-t^\nu)^{\xi+1}|^p t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\tau^\nu+\frac{1+t^\nu}{2}\kappa^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Big].$$

By using the convexity of $|f''|$, we have

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1 (1-t^\nu)^{p(\xi+1)}t^{\nu-1}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\nu-1}(\frac{1-t^\nu}{2})|f''(a^\nu)|^q dt$$

$$+\int_0^1 t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}+\Big(\int_0^1 (1-t^\nu)^{p(\xi+1)}t^{\nu-1}dt\Big)^{\frac{1}{p}}$$

$$\Big(\int_0^1 t^{\nu-1}(\frac{1-t^\nu}{2})|f''(\tau^\nu)|^q dt+\int_0^1 t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\kappa^\nu)|^q dt\Big)^{\frac{1}{q}}\Big].$$

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big[(\frac{1}{2\nu})|f''(\kappa^\nu)|^q+(\frac{3}{2\nu})|f''(\tau^\nu)|^q\Big]\Big)^{\frac{1}{q}}$$

$$\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big[(\frac{1}{2\nu})|f''(\tau^\nu)|^q+(\frac{3}{2\nu})|f''(\kappa^\nu)|^q\Big]\Big)^{\frac{1}{q}}\Big].$$

Namely,

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\frac{1}{p(\xi+1)+1}\Big)^{\frac{1}{p}}\Big[\Big(\frac{|f''(\kappa^\nu)|^q+3|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}+\Big(\frac{3|f''(\kappa^\nu)|^q+|f''(\tau^\nu)|^q}{4\nu}\Big)^{\frac{1}{q}}\Big]\Big].$$

This completes the proof. □

Theorem 2.11. *If $f:[\kappa^\nu,\tau^\nu]\rightarrow\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f''\in L_1[\kappa^\nu,\tau^\nu]$. If $|f''|$ is a concave function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\frac{1}{\nu(\xi+2)}\Big[f''\Bigg(\frac{\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)$$

$$+f''\Bigg(\frac{\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)\Big]\Big].$$

Proof. By applying Jensen inequality on R.H.S of lemma (2.2), we can write

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt$$

$$+\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|dt\Big]$$

$$\leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Big[\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)\Bigg|f''\Bigg(\frac{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)dt}{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt}\Bigg)\Bigg|$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)\Bigg|f''\Bigg(\frac{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)dt}{\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt}\Bigg)\Bigg|.$$

By a simple computation, one has

$$|B| \le \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\frac{1}{\nu(\xi+2)}\Bigg[f''\Bigg(\frac{\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)$$

$$+f''\Bigg(\frac{\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\frac{\kappa^\nu}{2}+\Big(\frac{1}{\nu(\xi+3)}\Big)\frac{\tau^\nu}{2}}{\frac{1}{\nu(\xi+2)}}\Bigg)\Bigg]\Bigg].$$

This is the desired result. □

Theorem 2.12. *If $f:[\kappa^\nu,\tau^\nu]\to\Re$ be differentiable function on (κ^ν,τ^ν) with $\kappa^\nu<\tau^\nu$ and $f''\in L_1[\kappa^\nu,\tau^\nu]$. If $|f''|^q$ is a convex function, then we have the following inequality for Katugampola fractional integral operators:*

$$\frac{2^{\xi-1}\Gamma(\xi+1)\nu^\xi}{(\tau^\nu-\kappa^\nu)^\xi}\Big[\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_+}\Big)f\big(\tau^\nu\big)+\Big({}^\nu I^\xi_{\left(\frac{\kappa^\nu+\tau^\nu}{2}\right)^{\frac{1}{\nu}}_-}\Big)f\big(\kappa^\nu\big)\Big]-f\Big(\frac{\kappa^\nu+\tau^\nu}{2}\Big)$$

$$\le \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(a^\nu)|^q}{2}\Big(\frac{2}{\nu(\xi+2)}\Big)+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{2}{\nu(\xi+2)}\Big)\Bigg]^{\frac{1}{q}}\Bigg].$$

Proof. By applying Power mean inequality on R.H.S of lemma (2.2), we have

$$|B| \le \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|dt$$

$$+\int_0^1 \Big((1-t^\nu)^{\xi+1}t^{\nu-1}\Big)\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|dt\Bigg]$$

$$\le \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}$$

$$\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1-t^\nu}{2}\kappa^\nu+\frac{1+t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}\Big|f''\Big(\frac{1+t^\nu}{2}\kappa^\nu+\frac{1-t^\nu}{2}\tau^\nu\Big)\Big|^q dt\Bigg)^{\frac{1}{q}}\Bigg]$$

By using convexity of $|f''|^q$, we get

$$|B| \le \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}(\frac{1-t^\nu}{2})|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\tau^\nu)|^q dt\Bigg)^{\frac{1}{q}}$$

$$+\Big(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}dt\Big)^{1-\frac{1}{q}}\Bigg(\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}(\frac{1+t^\nu}{2})|f''(\kappa^\nu)|^q dt$$

$$+\int_0^1 (1-t^\nu)^{\xi+1}t^{\nu-1}(\frac{1-t^\nu}{2})|f''(\tau^\nu)|^q dt\Big)^{\frac{1}{q}}\Bigg].$$

By computing the above integrals, we obtain

$$|B| \leq \frac{\nu(\tau^\nu-\kappa^\nu)^2}{8(\xi+1)}\Bigg[\Big(\frac{1}{\nu(\xi+2)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{|f''(\kappa^\nu)|^q}{2}\Big(\frac{1}{\nu(\xi+3)}\Big)+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)$$

$$+\frac{|f''(\tau^\nu)|^q}{2}\Big(\frac{1}{\nu(\xi+3)}\Big)+\frac{|f''(\kappa^\nu)|^q}{2}\Big(\frac{\xi+4}{\nu(\xi+2)(\xi+3)}\Big)\Bigg]^{\frac{1}{q}}\Bigg].$$

This is the desired result. □

3. Conclusion

In the literature, there are many studies of different researchers that include Katugampola integral operators for functions whose absolute values of first derivatives are convex. The main motivation point of the study is to obtain the inequalities with the help of Katugampola integral operators for the functions whose absolute value of the second derivatives are convex and concave functions. In this sense, the findings contribute to the improvement in convex analysis and take the discussion one step further. In addition, Hölder's inequality is used to prove the main results and new approaches are obtained.
Recently, researchers working in the field of inequalities frequently use fractional integral operators and thus obtain new generalizations associated with the certain types of inequalities. Katugampola integral operators structurally combine Riemann-Liouville and Hadamard fractional integral operators and contribute to the effectiveness of the results with its generalized kernel structure.
The results can be performed for different kinds of convexity and operators.These results can be applied in convex analysis, optimization and different areas of pure and applied sciences. The authors hope that these results will serve as a motivation for future work in this fascinating area.

Acknowledgement

The research of the first author has been fully supported by H.E.C. Pakistan under NRPU project 7906.

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