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Coefficient Bound Estimates and Fekete-Szegö Problem for a Certain Subclass of Analytic and Bi-univalent Functions

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Abstract. In this study, we introduce and examine a certain subclass of analytic and bi-univalent functions in the open unit disk in the complex plane. Here, we give coefficient bound estimates and examine the Fekete-Szegö problem for this class. Some interesting special cases of the results obtained here are also discussed.

1. Introduction and preliminaries

Let A denote the class of all complex valued functions $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z + a_2z^2 + a_3z^3 + ... + a_nz^n + ... = z + \sum_{n=2}^{\infty} a_nz^n,\ z \in \mathbb{C}, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. By S, we will denote the class of all univalent functions in the set A. For $\alpha \in [0, 1)$, some of the important and well-investigated subclasses of S include the classes $S^* (\alpha)$ and $C (\alpha)$, respectively, starlike and convex function classes of order α in U.

It is well-known that (see [3]) every function $f \in S$ has an inverse f^{-1} defined by

$$f^{-1} (f(z)) = z, z \in U,\ f^{-1} (f(w)) = w, w \in U_0 = \{w \in \mathbb{C} : |w| < r_0 (f)\}, r_0 (f) \geq \frac{1}{4}$$

and

$$f^{-1}(w) = w + b_2w^2 + b_3w^3 + ... + b_nw^n + ... = w + \sum_{n=2}^{\infty} b_nw^n,\ w \in U_0,$$

where

$$b_2 = -a_2,\ b_3 = 2a_2^2 - a_3,\ b_4 = -5a_2^3 + 5a_2a_3 - a_4.$$

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A function $f \in A$ is called bi-univalent in U if both f and f^{-1} are univalent in U and $f(U)$ respectively. Let Σ denote the class of bi-univalent functions in the set S.

For the functions f and g which are analytic in U, f is said to be subordinate to g and denoted as $f(z) \prec g(z)$ if there exists an analytic function ω such that

$$\omega(0) = 0, |\omega(z)| < 1 \text{ and } f(z) = g(w(z)).$$

As is known that the coefficient problem is one of the important subjects of the theory of geometric functions. Firstly, by Lewin was introduced [7] a subclass of bi-univalent functions and obtained the estimate $|a_2| \leq 1.51$ for the function belonging to this class. Subsequently, Brannan and Clunie [1] developed the result of Lewin to $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [11] showed that $|a_2| \leq \frac{4}{3}$ for this class functions. By Brannan and Taha [2] were introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike function of order α denoted $S^*_\Sigma(\alpha)$ and bi-convex function of order α denoted $C_\Sigma(\alpha)$, respectively. For each of the function classes $S^*_\Sigma(\alpha)$ and $C_\Sigma(\alpha)$, non-sharp estimates on the first two coefficients for the functions belonging to these classes were found by Brannan and Taha (see [2]). Many researchers have introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found non-sharp estimates on the first two coefficients for the functions belonging to these classes (see [13, 15]).

It is also well known that the important tools in the theory of analytic functions is the functional $H_2(1) = a_3 - a_2^2$, which is known as the Fekete-Szegö functional and one usually considers the further generalized functional $H_2(1) = a_3 - \mu a_2^2$, where μ is a complex or real number (see [5]). Estimating the upper bound of $\left|a_3 - \mu a_2^2\right|$ is known as the Fekete-Szegö problem in the theory of analytic functions. The Fekete-Szegö problem has been investigated by many mathematicians for several subclasses of analytic functions (see [8, 9, 14]). Very soon, Mustafa and Mrugusundaramoorthy [10] examine the Fekete-Szegö problem for the subclass of bi-univalent functions related to shell shaped region.

Now, let's give some concepts that we will use throughout our study.

For $q \in (0, 1)$, in his fundamental paper by Jackson [6] introduced $q-$derivative operator D_q of an analytic function f as follows:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & if \quad z \neq 0, \\ f' & if \quad z = 0. \end{cases} \tag{2}$$

It follows from that $D_q z^n = [n]_q z^{n-1}, n \in \mathbb{N}$, where $[n]_q = 1 + q + q^2 + ... + q^{n-1} = \sum\limits_{k=1}^{n} q^{k-1}$ is $q-$analogue of the natural numbers n. Also, it can be easily shown that $\lim\limits_{q \to 1^{-1}} [n]_q = n$, $[n]_q \frac{1-q^n}{1-q}, [0]_q = 0, [1]_q = 1$.

Using definition (2) for the first and second $q-$ derivative of the function $f \in A$, we write

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q z^{n-1} \text{ and } D_q^2 f(z) = D_q\left(D_q f(z)\right) = \sum_{n=2}^{\infty} [n]_q [n-1]_q z^{n-2}.$$

Also, it is clear that $\lim\limits_{q \to 1^-} D_q f(z) = f'(z)$ for an analytic function f.

For the function $f \in A$, Salagean (see [12]) introduced the following differential operator, which is called the Salagean operator

$$S^0 f(z) = f(z), S^1 f(z) = zSf(z) = zf'(z),$$
$$S^2 f(z) = zS(Sf(z)) = zf''(z), ..., S^n f(z) = zS\left(S^{n-1} f(z)\right), n = 1, 2, ...$$

It follows from that

$$S^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now, let we define the following subclass of analytic and bi-univalent functions.

Definition 1.1. *For $q \in (0,1)$, a function $f \in \Sigma$ is said to be in the class $C_{q,\Sigma}(n,\varphi)$ if the following conditions are satisfied*

$$1+\frac{zD_q^2\left(S^n f(z)\right)}{D_q\left(S^n f(z)\right)} \prec \varphi(z), z \in U \text{ and } 1+\frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q\left(S^n f^{-1}(w)\right)} \prec \varphi(w), w \in U_0.$$

In this definition $\varphi(z) = z + \sqrt{1+z^2}$ and the branch of the square root is chosen to be principal one, that $\varphi(0) = 1$. It can be easily seen that the function $\varphi(z) = z + \sqrt{1+z^2}$ maps the unit disc U onto a shell shaped region on the right half plane and it is analytic and univalent in U. The range $\varphi(U)$ is symmetric respect to real axis and φ is a function with positive real part in U, with $\varphi(0) = \varphi'(0) = 1$ Moreover, it is a starlike domain with respect to point $\varphi(0) = 1$.

In the case $n = 0$, from the Definition 1.1 we have the subclass $C_{q,\Sigma}(\varphi) = C_{q,\Sigma}(0,\varphi)$. Also, we have the subclass $C_\Sigma(n,\varphi)$, when $q \to 1^-$.

Let, P be the set of the functions $p(z)$ analytic in U and satisfying $\Re(p(z)) > 0$, $z \in U$ and $p(0) = 1$ with power series

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... + p_n z^n + ... = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in U,$$

In order to prove our main results in this paper, we shall need the following lemmas (see [3, 4]).

Lemma 1.2. *Let $p \in P$, then $\left|p_n\right| \le 2$, $n = 1,2,3,....$ These inequalities are sharp. In particular, equality holds for the function $p(z) = (1+z)/(1-z)$ for all $n = 1,2,3,....$*

Lemma 1.3. *Let $p \in P$, then $\left|p_n\right| \le 2$, $n = 1,2,3,...$ and*

$$2p_2 = p_1^2 + \left(4-p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4-p_1^2\right)p_1 x - 2\left(4-p_1^2\right)p_1 x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

for some x and z with $|x| < 1$ and $|z| < 1$.

Remark 1.4. *As can be seen from the serial expansion of the function φ given in Definition 1.1, this function belong to the class P.*

In this paper, we give coefficient bound estimates and examine the Fekete-Szegö problem for the class $C_{q,\Sigma}(n,\varphi)$.

2. Main Results

In this section, firstly we give the following theorem on the coefficient bound estimates for the class $C_{q,\Sigma}(n,\varphi)$.

Theorem 2.1. *Let the function f given by (1) be in the class $C_{q,\Sigma}(n,\varphi)$. Then*

$$|a_2| \le \frac{1}{[2]_q 2^n}, |a_3| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n}, & \frac{[3]_q}{[2]_q} \le \left(\frac{4}{3}\right)^n, \\ \frac{1}{[2]_q^2 4^n} & \frac{[3]_q}{[2]_q} > \left(\frac{4}{3}\right)^n. \end{cases}$$

Moreover,

$$|a_4| \leq \begin{cases} \max\{\lambda(q,n), \nu(q,n)\}, & \theta_1(q,n) \geq 0, \\ \max\left\{\lambda(q,n), \nu(q,n), \sqrt{\frac{-4\theta_2^3(q,n)}{27\theta_1(q,n)}}\right\}, & \theta_1(q,n) < 0, \end{cases}$$

where

$$\lambda(q,n) = \frac{\left([2]_q+1\right)[3]_q 3^n - [2]_q^2 4^n}{[2]_q^2 [3]_q [4]_q 16^n}, \nu(q,n) = \frac{1}{[3]_q [4]_q 4^n},$$

$$\theta_1(q,n) = \frac{\left([2]_q+1\right)[3]_q 3^n - [2]_q^2 4^n}{8[2]_q^2 [3]_q [4]_q 16^n} - \frac{5}{16[2]_q^2 [3]_q 6^n} - \frac{1}{[3]_q [4]_q 4^n},$$

$$\theta_2(q,n) = \frac{5}{4[2]_q^2 [3]_q 6^n} + \frac{1}{[3]_q [4]_q 4^{n-1}}.$$

Proof. Let $f \in C_{q,\Sigma}(n,\varphi)$. Then, according to Definition 1.1 there are analytic functions $\omega : U \to U$ and $\varpi : U_0 \to U_0$ with $\omega(0) = 0 = \varpi(0), |\omega(z)| < 1$ and $|\varpi(z)| < 1$ satisfying the following conditions

$$\begin{aligned} 1 + \frac{zD_q^2(S^n f(z))}{D_q(S^n f(z))} &= \varphi(\omega(z)) = \omega(z) + \sqrt{1+\omega^2(z)},\ z \in U, \\ 1 + \frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q(S^n f^{-1}(w))} &= \varphi(\varpi(w)) = \varpi(w) + \sqrt{1+\varpi^2(w)},\ w \in U_0. \end{aligned} \tag{3}$$

Now, we define the functions $p, \phi \in P$ as follows:

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... + p_n z^n + ... = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in U$$

$$\phi(w) = \frac{1+\varpi(z)}{1-\varpi(z)} = 1 + \phi_1 w + \phi_2 w^2 + \phi_3 w^3 + ... + \phi_n w^n + ... = 1 + \sum_{n=1}^{\infty} \phi_n w^n,\ w \in U_0.$$

From here, we find the following equalities for the functions ω and ϖ

$$\begin{aligned} \omega(z) &= \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^2}{4}\right)z^3 + ...\right],\ z \in U, \\ \varpi(w) &= \frac{\phi(w)-1}{\phi(w)+1} = \frac{1}{2}\left[\phi_1 w + \left(\phi_2 - \frac{\phi_1^2}{2}\right)w^2 + \left(\phi_3 - \phi_1\phi_2 + \frac{\phi_1^2}{4}\right)w^3 + ...\right],\ w \in U_0, \end{aligned} \tag{4}$$

Changing the expression of the functions $\omega(z)$ and $\varpi(z)$ in (3) with expressions in (4), we can write the following equalities

$$\begin{aligned} & 1 + \frac{zD_q^2(S^n f(z))}{D_q(S^n f(z))} \\ =\ & 1 + \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4}\right)z^3 + ..., z \in U, \\ & 1 + \frac{zD_q^2\left(S^n f^{-1}(w)\right)}{D_q(S^n f^{-1}(w))} \\ =\ & 1 + \frac{\phi_1}{2}w + \left(\frac{\phi_2}{2} - \frac{\phi_1^2}{8}\right)w^2 + \left(\frac{\phi_3}{2} - \frac{\phi_1\phi_2}{4}\right)w^3 + ..., w \in U_0. \end{aligned} \tag{5}$$

If the operations and simplifications on the left side of (5) are made and then the coefficients of the terms of the same degree are equalized, are obtained the following equalities for the coefficients a_2, a_3 and a_4

$$[2]_q 2^n a_2 = \frac{p_1}{2}, \; [2]_q [3]_q 3^n a_3 - [2]_q^2 4^n a_2^2 = \frac{p_2}{2} - \frac{p_1^2}{8}$$

$$[3]_q [4]_q 4^n a_4 - [2]_q [3]_q \left([2]_q + 1\right) 6^n a_2 a_3 + [2]_q^2 8^n a_2^3 = \frac{p_3}{2} - \frac{p_1 p_2}{4}$$

and

$$-[2]_q 2^n a_2 = \frac{\phi_1}{2}, -[2]_q [3]_q 3^n a_3 + \left\{2 [2]_q [3]_q 3^n - [2]_q^2 4^n\right\} a_2^2 = \frac{\phi_2}{2} - \frac{\phi_1^2}{8}$$

$$\begin{aligned} & -[3]_q [4]_q 4^n a_4 + \left[5 [3]_q [4]_q 4^n - [2]_q [3]_q \left([2]_q + 1\right) 6^n\right] a_2 a_3 \\ & -\left[5 [3]_q [4]_q 4^n - 2 [2]_q [3]_q \left([2]_q + 1\right) 6^n + [2]_q^2 8^n\right] a_2^3 \\ = \; & \frac{\phi_3}{2} - \frac{\phi_1 \phi_2}{4}. \end{aligned}$$

From these equalities, we write

$$\frac{p_1}{[2]_q 2^{n+1}} = a_2 = -\frac{\phi_1}{[2]_q 2^{n+1}}, \; p_1 = -\phi_1, \tag{6}$$

$$a_3 = a_2^2 + \frac{p_2 - \phi_2}{4 [2]_q [3]_q 3^n}, \tag{7}$$

$$a_4 = \frac{5\left(p_2 - \phi_2\right)}{16 [2]_q^2 [3]_q 6^n} + \frac{[3]_q \left([2]_q + 1\right) 3^n - [2]_q^2 4^n}{[2]_q^2 [3]_q [4]_q 2^{4n+3}} p_1^3 + \frac{p_3 - \phi_3}{[3]_q [4]_q 4^{n+1}} - \frac{\left(p_2 + \phi_2\right) p_3}{2 [3]_q [4]_q 4^{n+1}}. \tag{8}$$

By applying the Lemma 1.2 to equality (6), obtained immediately first result of theorem.

Now, firstly using the Lemma 1.3 and then applying triangle inequality and Lemma 1.2 to the equality (7), we get

$$|a_3| = \frac{t^2}{[2]_q 4^{n+1}} + \frac{4 - t^2}{8 [2]_q [3]_q 3^n} (\xi + \eta).$$

with $\left|p_1\right| = t$, $|x| = \xi$ and $\left|y\right| = \eta$ for some x and y with $|x| < 1$ and $\left|y\right| < 1$. Then, maximizing the right-hand side of the last inequality according to the parameters $\xi \in (0, 1)$ and $\eta \in (0, 1)$, we obtain the following inequality

$$|a_3| \le c(q, n) t^2 + \frac{1}{[2]_q [3]_q 3^n}, t \in [0, 2], c(q, n) = \frac{[3]_q 3^n - [2]_q 4^n}{4 [2]_q^2 [3]_q 12^n}.$$

From the last inequality obtained the second result of theorem.

Finally, let's find an upper bound estimate for $|a_4|$. By applying Lemma 1.3 and then triangle inequality and Lemma 1.2 to expression of a_4 in the equality (8), we obtain

$$8 |a_4| \le c_1(t) + c_2(t)(\xi + \eta) + c_3(t)\left(\xi^2 + \eta^2\right), \tag{9}$$

with $|x| = \xi \in (0, 1)$ and $\left|y\right| = \eta \in (0, 1)$, where

$$c_1(t) = \frac{[3]_q\left([2]_q+1\right)3^n - [2]_q^2 4^n}{[2]_q^2 [3]_q [4]_q 2^{4n+3}} t^3 + \frac{1}{[3]_q [4]_q 4^{n+1}}\left(4-t^2\right),$$

$$c_2(t) = \left[\frac{5}{32[2]_q^2[3]_q 6^n} + \frac{1}{[3]_q[4]_q 4^{n+1}}\right]\left(4-t^2\right)t,\ c_3(t) = \frac{\left(4-t^2\right)(t-2)}{[3]_q[4]_q 4^{n+2}}$$

If we maximize the right-hand side of the inequality (9) firstly according to the parametres ξ and η and then to the parameter, we get the desired estmate for $|a_4|$.

Thus, the proof of Theorem 2.1 is completed. □

From the Teorem 2.1, we obtain the following results.

Corollary 2.2. *Let* $f \in C_\Sigma(n,\varphi)$. *Then,*

$$|a_2| \le \frac{1}{2^{n+1}}, n = 0,1,2,..., |a_3| \le \begin{cases} \frac{1}{4^{n+1}}, & if\ \ n = 0,1, \\ \frac{1}{2.3^{n+1}} & otherwise \end{cases}$$

and

$$|a_4| \le \max\left\{\left(\frac{3}{16}\right)^{n+1} - \frac{1}{3.4^{n+1}}, \frac{1}{3.4^{n+1}}, \sqrt{\frac{-4\theta_2^3(n)}{27\theta_1(n)}}\right\}$$

where

$$\theta_1(q,n) = \frac{3^{n+1}}{8.16^{n+1}} - \frac{5}{32.6^n} - \frac{3}{8.4^{n+1}},$$

$$\theta_2(q,n) = \frac{5}{8.6^{n+1}} + \frac{1}{3.4^n}.$$

Corollary 2.3. *Let* $C_{q,\Sigma}(\varphi)$. *Then*

$$|a_2| \le \frac{1}{q+1}, |a_3| \le \frac{1}{(q+1)^2}$$

Moreover,

$$|a_4| \le \max\left\{\lambda(q), \nu(q), \sqrt{\frac{-4\theta_2^3(q)}{27\theta_1(q)}}\right\},$$

where

$$\lambda(q) = \frac{q^3+2q^2+q+1}{(q+1)^2(q^2+q+1)(q^3+q^2+q+1)}, \nu(q) = \frac{1}{(q^2+q+1)(q^3+q^2+q+1)},$$

$$\theta_1(q) = \frac{-3q^3-17q^2-35q-19}{16(q+1)^2(q^2+q+1)(q^3+q^2+q+1)},$$

$$\theta_2(q,n) = \frac{5q^3+21q^2+37q+16}{4(q+1)^2(q^2+q+1)(q^3+q^2+q+1)}.$$

Now, we give the following theorem on the Fekete-Szegö problem for the class $C_{q,\Sigma}(n,\varphi)$.

Theorem 2.4. *Let the function f given by (1) be in the class $C_{q,\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n} & if \quad \left|1-\mu\right| \le \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n, \\ \frac{\left|1-\mu\right|}{[2]_q^2 4^n} & if \quad \left|1-\mu\right| > \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n. \end{cases}$$

Proof. Let $f \in C_{q,\Sigma}(n,\varphi)$ and $\mu \in \mathbb{C}$. Then, from the expressions for a_2 and a_3 in the equalities (6) and (7), we can write the following equality for $a_3-\mu a_2^2$

$$a_3-\mu a_2^2 = (1-\mu)\,a_2^2 + \frac{p_2-\phi_2}{4\,[2]_q\,[3]_q\,3^n}.$$

According to Lemma 1.3, from the last equality we can write

$$a_3-\mu a_2^2 = (1-\mu)\,a_2^2 + \frac{4-p_1^2}{8\,[2]_q\,[3]_q\,3^n}\,(x-y)$$

for some x and y with $|x|<1$ and $\left|y\right|<1$.

Then, using triangle inequality and considering that $\left|p_1\right| = t \le 2$ from the last equality, we get

$$\left|a_3-\mu a_2^2\right| \le \frac{(1-\mu)}{[2]_q^2\,4^{n+1}}t^2 + \frac{4-t^2}{8\,[2]_q\,[3]_q\,3^n}\,(\xi+\eta), \tag{10}$$

with $|x| = \xi \in (0,1)$ and $\left|y\right| = \eta \in (0,1)$. If we maximize the right-hand side of the inequality (10) according to the parametres ξ and η we get the following inequality

$$\left|a_3-\mu a_2^2\right| \le \frac{1}{[2]_q^2\,4^{n+1}}\left\{\left|1-\mu\right| - \frac{[2]_q}{[3]_q}\left(\frac{4}{3}\right)^n\right\}t^2 + \frac{1}{[2]_q\,[3]_q\,3^n},\, t\in[0,2].$$

From here, by maximizing the right hand side of the last inequality according to the parameter t, obtained the result of theorem. Thus, the proof of Theorem 2.4 is completed. □

From the Theorem 2.4 obtained the following results.

Corollary 2.5. *Let $f \in C_\Sigma(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{2.3^{n+1}} & if \quad \left|1-\mu\right| \le \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}, \\ \frac{\left|1-\mu\right|}{4^{n+1}} & if \quad \left|1-\mu\right| > \frac{1}{2}\left(\frac{4}{3}\right)^{n+1}. \end{cases}$$

Corollary 2.6. *Let $f \in C_{q,\Sigma}(\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q} & if \quad \left|1-\mu\right| \le \frac{[2]_q}{[3]_q}, \\ \frac{\left|1-\mu\right|}{[2]_q^2} & if \quad \left|1-\mu\right| > \frac{[2]_q}{[3]_q}. \end{cases}$$

Corollary 2.7. *Let $f \in C_{q,\Sigma}(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{[2]_q[3]_q 3^n} & if \quad \frac{[3]_q}{[2]_q} \le \left(\frac{4}{3}\right)^n, \\ \frac{\left|1-\mu\right|}{[2]_q^2 4^n} & if \quad \frac{[3]_q}{[2]_q} > \left(\frac{4}{3}\right)^n. \end{cases}$$

Corollary 2.8. *Let $f \in C_\Sigma(n,\varphi)$. Then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{1}{4^{n+1}} & if \quad n=0,1, \\ \frac{1}{2.3^{n+1}} & otherwise. \end{cases}$$

Corollary 2.9. *Let $f \in C_{q,\Sigma}(\varphi)$. Then*

$$|a_3| \leq \frac{1}{[2]_q^2}.$$

Remark 2.10. *The Corollary 2.7 confirm the second result obtained in the Theorem 2.1.*

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