TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 3, 231–244
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Exponentially $m-$ and $(\alpha, m)-$Convex Functions on the Coordinates and Related Inequalities

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Abstract. In the present paper, new classes of convexity, namely, exponentially $m-$ and$(\alpha, m)-$convex functions on the co-ordinates are defined. Then, some new integral inequalities are proved by using some classical inequalities and properties of exponentially $m-$ ad $(\alpha, m)-$convex functions on the co-ordinates.

1. Introduction

In [8], Toader defined $m-$convex functions as following:

Definition 1.1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m(b)$ the class of all $m-$convex functions on $[0, b]$ for which $f(0) \leq 0$. Obviously, if we choose $m = 1$, we have ordinary convex functions on $[0, b]$.

In [7], Miheşan introduced $(\alpha, m)-$convexity as following:

Definition 1.2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$.

For several results related to above definitions we refer interest of readers to [4], [5], [6], [7], [8], [9] and [11].

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Received: 14 October 2022; Accepted: 10 December 2022; Published: 30 December 2022

Keywords. coordinates, exponentially convex functions. Hölder inequality.

2010 *Mathematics Subject Classification*. 26D15, 26A51

Cited this article as: Aslan S. Akdemir A.O. Dokuyucu M.A. Exponentially $m-$ and $(\alpha, m)-$Convex Functions on the Coordinates and Related Inequalities, Turkish Journal of Science, 2022, 7(3), 231–244.

We will start by expressing an important inequality proved for convex functions. This inequality is presented on the basis of averages and give bounds for the mean value of a convex function.
Assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.
In [1], Dragomir mentions an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1.3. *Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall that the mapping $f : \Delta \to \mathbb{R}$ is convex on Δ if the following inequality holds,*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [1]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$ stated below.

Theorem 1.4. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (1)\\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx \right.\\
& \left. + \frac{1}{(d-c)}\int_c^d f(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

The concept of exponentially convex function on the coordinates and the associated results are presented as the followings:

Definition 1.5. *(See [12]) Let us consider the interval such as $\Delta = [\epsilon_1, \epsilon_2] \times [\epsilon_3, \epsilon_4]$ in $\mathbb{R}^2$ with $\epsilon_1 < \epsilon_2$, $\epsilon_3 < \epsilon_4$. The function $\Psi : \Delta \to \mathbb{R}$ is exponentially convex on Δ if*

$$\Psi((1-\zeta)u_1 + \zeta u_3, (1-\zeta)u_2 + \zeta u_4) \leq (1-\zeta)\frac{\Psi(u_1,u_2)}{e^{\alpha(u_1+u_2)}} + \zeta\frac{\Psi(u_3,u_4)}{e^{\alpha(u_3+u_4)}}$$

for all $(u_1,u_2),(u_3,u_4) \in \Delta$, $\alpha \in \mathbb{R}$ and $\zeta \in [0,1]$.

An equivalent definition of the exponentially convex function definition in coordinates can be done as follows:

Definition 1.6. *(See [12]) The mapping $\Psi : \Delta \longrightarrow \mathbb{R}$ is exponentially convex function on the co-ordinates on Δ, if*

$$\begin{aligned} &\Psi\left(\zeta\epsilon_1 + (1-\zeta)\epsilon_2, \xi\epsilon_3 + (1-\xi)\epsilon_4\right) \\ \leq\ & \zeta\xi\frac{\Psi(\epsilon_1,\epsilon_3)}{e^{\alpha(\epsilon_1+\epsilon_3)}} + \zeta(1-\xi)\frac{\Psi(\epsilon_1,\epsilon_4)}{e^{\alpha(\epsilon_1+\epsilon_4)}} + (1-\zeta)\xi\frac{\Psi(\epsilon_2,\epsilon_3)}{e^{\alpha(\epsilon_2+\epsilon_3)}} + (1-\zeta)(1-\xi)\frac{\Psi(\epsilon_2,\epsilon_4)}{e^{\alpha(\epsilon_2+\epsilon_4)}} \end{aligned}$$

for all $(\epsilon_1,\epsilon_3), (\epsilon_1,\epsilon_4), (\epsilon_2,\epsilon_3), (\epsilon_2,\epsilon_4) \in \Delta, \alpha \in \mathbb{R}$ and $\zeta, \xi \in [0,1]$.

The main motivation of this paper is to define exponentially $m-$ and $(\alpha, m)-$convex functions on the co-ordinates. We have proved several integral inequalities for these classes of functions.

2. Exponentially $m-$convex functions on the co-ordinates

Definition 2.1. *Let us consider the bidimensional interval $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponentially $m-$convex function on the co-ordinates on Δ, if the following inequality holds,*

$$f\left(tx + (1-t)z, ty + m(1-t)w\right) \leq t\frac{f(x,y)}{e^{\alpha(x+y)}} + m(1-t)\frac{f(z,w)}{e^{\alpha(z+w)}}.$$

for all $(x,y), (z,w) \in \Delta$,$\alpha \in R, m \in (0,1]$ and $t \in [0,1]$.

An equivalent definition of the exponentially $m-$convex function definition in coordinates can be done as follows:

Definition 2.2. *The mapping $f : \Delta \longrightarrow R$ is exponential convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} &f\left(ta + (1-t)b, sc + m(1-s)d\right) \\ \leq\ & ts\frac{f(a,c)}{e^{\alpha(a+c)}} + t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a,c), (a,d), (b,c), (b,d) \in \Delta, \alpha \in R$ and $m, t, s \in [0,1]$

Lemma 2.3. *A function $f : \Delta \longrightarrow R$ will be called exponential $m-$convex function on the co-ordinates on Δ, if the partial mappings $f_y : [a,b] \longrightarrow R$, $f_y(u) = e^{\alpha y} f(u,y)$ and $f_x : [c,d] \longrightarrow R$, $f_x(v) = e^{\alpha x} f(x,v)$ are exponentially $m-$convex function on the co-ordinates on Δ, where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Proof. From the definition of partial mapping f_x, we can write

$$\begin{aligned} f_x(tv_1 + m(1-t)v_2) &= e^{\alpha x} f(x, tv_1 + m(1-t)v_2) \\ &= e^{\alpha x} f(tx + (1-t)x, tv_1 + m(1-t)v_2) \\ &\leq e^{\alpha x}\left[t\frac{f(x,v_1)}{e^{\alpha(x+v_1)}} + m(1-t)\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\ &= t\frac{f(x,v_1)}{e^{\alpha v_1}} + m(1-t)\frac{f(x,v_2)}{e^{\alpha v_2}} \\ &= t\frac{f_x(v_1)}{e^{\alpha v_1}} + m(1-t)\frac{f_x(v_2)}{e^{\alpha v_2}} \end{aligned}$$

Similarly,

$$\begin{aligned}
f_y(tu_1 + m(1-t)u_2) &= e^{\alpha y} f(tu_1 + m(1-t)u_2, y) \\
&= e^{\alpha y} f(tu_1 + m(1-t)u_2, ty + (1-t)y) \\
&\leq e^{\alpha y}\left[t\frac{f(u_1, y)}{e^{\alpha(u_1+y)}} + m(1-t)\frac{f(u_2, y)}{e^{\alpha(u_2+y)}}\right] \\
&= t\frac{f(u_1, y)}{e^{\alpha u_1}} + m(1-t)\frac{f(u_2, y)}{e^{\alpha u_2}} \\
&= t\frac{f_y(u_1)}{e^{\alpha u_1}} + m(1-t)\frac{f_y(u_2)}{e^{\alpha u_2}}.
\end{aligned}$$

The proof is completed. □

Theorem 2.4. *Let $f : \Delta = [0, b] \times [0, d] \to R$ be partial differentiable mapping on $\Delta = [0, b] \times [0, d]$ in R^2 with $0 < a < b < \infty$ and $0 < c < md < \infty$, $f \in L(\Delta)$, $\alpha \in R$. If f is exponentially $m-$convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}
&\frac{1}{(b-a)(md-c)}\int_a^b \int_c^{md} f(x, y)dxdy \\
\leq\ & \frac{1}{4}\left[\frac{f(a, c)}{e^{\alpha(a+c)}} + \frac{f(b, c)}{e^{\alpha(b+c)}} + m\left(\frac{f(b, d)}{e^{\alpha(b+d)}} + \frac{f(a, d)}{e^{\alpha(a+d)}}\right)\right].
\end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f(ta + (1-t)b, sc + m(1-s)d) \\
\leq\ & ts\frac{f(a, c)}{e^{\alpha(a+c)}} + mt(1-s)\frac{f(a, d)}{e^{\alpha(a+d)}} + (1-t)s\frac{f(b, c)}{e^{\alpha(b+c)}} + m(1-t)(1-s)\frac{f(b, d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

By integrating both sides of the above inequality with respect to t, s on $[0, 1]^2$, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 f(ta + (1-t)b, sc + m(1-s)d)\, dtds \\
\leq\ & \int_0^1 \int_0^1 ts\frac{f(a, c)}{e^{\alpha(a+c)}}dtds + \int_0^1 \int_0^1 t(1-s)m\frac{f(a, d)}{e^{\alpha(a+d)}}dtds \\
& + \int_0^1 \int_0^1 (1-t)s\frac{f(b, c)}{e^{\alpha(b+c)}}dtds + \int_0^1 \int_0^1 (1-t)(1-s)m\frac{f(b, d)}{e^{\alpha(b+d)}}dtds.
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.5. *Let $f : \Delta = [0, b] \times [0, d] \to R$ be partial differentiable mapping on $\Delta = [0, b] \times [0, d]$ in R^2 with $0 < a < b < \infty$ and $0 < c < md < \infty$, $f \in L(\Delta)$, $\alpha \in R$. If $|f|$ is exponentially $m-$convex function on the co-ordinates on Δ, $p > 1$ and $m \in (0, 1]$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b \int_c^{md} f(x, y)dxdy\right| \\
\leq\ & \left(\frac{1}{(p+1)^2}\right)^{\frac{1}{p}}\left(\frac{|f(a, c)|}{e^{\alpha(a+c)}} + \frac{|mf(a, d)|}{e^{\alpha(a+d)}} + \frac{|f(b, c)|}{e^{\alpha(b+c)}} + \frac{|mf(b, d)|}{e^{\alpha(b+d)}}\right).
\end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f\left(ta+(1-t)b, sc+m(1-s)d\right) \\
\leq\; & ts\frac{f(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}+ \\
& (1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}
& \left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\
\leq\; & \int_0^1\int_0^1\left|ts\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
& +\int_0^1\int_0^1\left|(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
& \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\; & \left(\int_0^1\int_0^1 t^p s^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 t^p(1-s)^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^p s^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\int_0^1 (1-t)^p(1-s)^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 2.6. *Let $f:\Delta=[0,b]\times[0,d]\to R$ be partial differentiable mapping on $\Delta=[0,b]\times[0,d]$ in R^2 with $0<a<b<\infty$ and $0<c<md<\infty$, $f\in L(\Delta)$, $\alpha\in R$. If $\left|f\right|$ is exponentially $m-$convex function on the co-ordinates on Δ, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds;*

$$\begin{aligned}
& \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\; & \left(\frac{4}{p(p+1)^2}\right) \\
& +\frac{1}{q}\left(\frac{\left|f(a,c)\right|^q}{e^{\alpha q(a+c)}}+\frac{\left|mf(a,d)\right|^q}{e^{\alpha q(a+d)}}+\frac{\left|f(b,c)\right|^q}{e^{\alpha q(b+c)}}+\frac{\left|mf(b,d)\right|^q}{e^{\alpha q(b+d)}}\right).
\end{aligned}$$

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \ & ts\frac{f(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

By the absolute value property and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned} & \left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\ \leq \ & \int_0^1\int_0^1\left|ts\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\ & +\int_0^1\int_0^1\left|(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds \end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned} & \left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\ \leq \ & \frac{1}{p}\left(\int_0^1\int_0^1 t^ps^pdtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 t^p(1-s)^pdtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^ps^pdtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\ & +\frac{1}{p}\left(\int_0^1\int_0^1 (1-t)^p(1-s)^pdtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right). \end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Proposition 2.7. *If $f,g:\Delta\rightarrow R$ are two exponentially $m-$convex functions on the co-ordinates on Δ, then $f+g$ is exponentially $m-$convex function on the co-ordinates on Δ.*

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right)+g\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \ & ts\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right)+t(1-s)m\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\ & +(1-t)s\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right)+(1-t)(1-s)m\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right). \end{aligned}$$

Namely,

$$\begin{aligned} & (f+g)\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \ & ts\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{(f+g)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{(f+g)(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore, $(f+g)$ is exponentially $m-$convex functions on the co-ordinates on Δ. □

Proposition 2.8. *If $f : \Delta \to R$ is exponential $m-$convex functions on the co-ordinates on Δ and $k \geq 0$ then kf is exponential $m-$convex functions on the co-ordinates on Δ.*

Proof. By the definition of the exponentially $m-$convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \;& ts\frac{f(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have

$$\begin{aligned} & (kf)\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \;& ts\frac{(kf)(a,c)}{e^{\alpha(a+c)}}+t(1-s)m\frac{(kf)(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t)s\frac{(kf)(b,c)}{e^{\alpha(b+c)}}+(1-t)(1-s)m\frac{(kf)(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponentially $m-$convex functions on the co-ordinates on Δ. □

3. Exponentially $(\alpha, m)-$convex functions on the co-ordinates

Definition 3.1. *Let us consider the bidimensional interval $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty$ and $c < d$. The mapping $f : \Delta \longrightarrow R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, if the following inequality holds,*

$$f\left(tx+(1-t)z, ty+m(1-t)w\right) \leq t^{\alpha_1}\frac{f(x,y)}{e^{\alpha(x+y)}}+m(1-t^{\alpha_1})\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y),(z,w) \in \Delta, \alpha \in R, (\alpha_1, m) \in [0,1]^2$ and $t \in [0,1]$.

An equivalent definition of the exponentially (α_1, m)-convex function definition in coordinates can be done as follows:

Definition 3.2. *The mapping $f : \Delta \longrightarrow R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} & f\left(ta+(1-t)b, sc+m(1-s)d\right) \\ \leq \;& t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}} \end{aligned}$$

for all $(a,c),(a,d),(b,c),(b,d) \in \Delta, \alpha \in R, (\alpha_1, m) \in [0,1]^2$ and $t, s \in [0,1]$

Lemma 3.3. *A function $f : \Delta \longrightarrow R$ will be called exponentially (α_1, m)-convex on the co-ordinates on Δ, if the partial mappings $f_y : [a,b] \longrightarrow R$, $f_y(u) = e^{\alpha y}f(u,y)$ and $f_x : [c,d] \longrightarrow R$, $f_x(v) = e^{\alpha x}f(x,v)$ are exponentially (α_1, m)-convex on the co-ordinates on Δ, where defined for all $y \in [c,d]$ and $x \in [a,b]$.*

Proof. From the definition of partial mapping f_x, we can write

$$\begin{aligned}
f_x(tv_1 + m(1-t)v_2) &= e^{\alpha x} f(x, tv_1 + m(1-t)v_2) \\
&= e^{\alpha x} f(tx + (1-t)x, tv_1 + m(1-t)v_2) \\
&\le e^{\alpha x}\left[t^{\alpha_1}\frac{f(x,v_1)}{e^{\alpha(x+v_1)}} + m(1-t^{\alpha_1})\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\
&= t^{\alpha_1}\frac{f(x,v_1)}{e^{\alpha v_1}} + m(1-t^{\alpha_1})\frac{f(x,v_2)}{e^{\alpha v_2}} \\
&= t^{\alpha_1}\frac{f_x(v_1)}{e^{\alpha v_1}} + m(1-t^{\alpha_1})\frac{f_x(v_2)}{e^{\alpha v_2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_y(tu_1 + m(1-t)u_2) &= e^{\alpha y} f(tu_1 + m(1-t)u_2, y) \\
&= e^{\alpha y} f(tu_1 + m(1-t)u_2, ty + (1-t)y) \\
&\le e^{\alpha y}\left[t^{\alpha_1}\frac{f(u_1,y)}{e^{\alpha(u_1+y)}} + m(1-t^{\alpha_1})\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right] \\
&= t^{\alpha_1}\frac{f(u_1,y)}{e^{\alpha u_1}} + m(1-t^{\alpha_1})\frac{f(u_2,y)}{e^{\alpha u_2}} \\
&= t^{\alpha_1}\frac{f_y(u_1)}{e^{\alpha u_1}} + m(1-t^{\alpha_1})\frac{f_y(u_2)}{e^{\alpha u_2}}.
\end{aligned}$$

The proof is completed. □

Theorem 3.4. *Let $f : \Delta = [0,b] \times [0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b] \times [0,d]$ in R^2 with $0 < a < b < \infty, 0 < c < md < \infty, f \in L(\Delta), (\alpha_1, m) \in [0,1]^2$ and $\alpha \in R$. If f is exponentially $(\alpha_1, m)-$convex function on the co-ordinates on Δ, then the following inequality holds;*

$$\begin{aligned}
&\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy \\
\le\ &\frac{1}{(\alpha_1+1)^2}\frac{f(a,c)}{e^{\alpha(a+c)}} + \frac{\alpha_1}{(\alpha_1+1)^2}\left(\frac{mf(a,d)}{e^{\alpha(a+d)}} + \frac{f(b,c)}{e^{\alpha(b+c)}}\right) \\
&+\frac{\alpha_1^2}{(\alpha_1+1)^2}\frac{mf(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

Proof. By the definition of the exponentially (α_1, m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
&f(ta + (1-t)b, sc + m(1-s)d) \\
\le\ &t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}} + t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\
&+(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}} + (1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

By integrating both sides of the above inequality with respect to t, s on $[0,1]^2$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 f(ta + (1-t)b, sc + m(1-s)d)\,dtds \\
\le\ &\int_0^1\int_0^1 t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}dtds + \int_0^1\int_0^1 t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}}dtds \\
&+\int_0^1\int_0^1 (1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}dtds + \int_0^1\int_0^1 (1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}dtds.
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 3.5. *Let $f:\Delta=[0,b]\times[0,d]\to R$ be partial differentiable mapping on $\Delta=[0,b]\times[0,d]$ in R^2 with $0<a<b<\infty, 0<c<md<\infty, f\in L(\Delta), (\alpha_1,m)\in[0,1]^2$ and $\alpha\in R$. If $\left|f\right|$ is exponentially (α_1,m)-convex on the co-ordinates on Δ, $p>1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \left(\frac{1}{(p\alpha_1+1)^2}\right)^{\frac{1}{p}}\frac{\left|f(a,c)\right|}{e^{\alpha(a+c)}} \\
&+\left(\frac{p\alpha_1}{(p\alpha_1+1)^2}\right)^{\frac{1}{p}}\left(\frac{m\left|f(a,d)\right|}{e^{\alpha(a+d)}}+\frac{\left|f(b,c)\right|}{e^{\alpha(b+c)}}\right) \\
&+\left(\frac{p^2\alpha_1^2}{(p\alpha_1+1)^2}\right)^{\frac{1}{p}}\frac{m\left|f(b,d)\right|}{e^{\alpha(b+d)}}.
\end{aligned}$$

Proof. By the definition of the exponentially (α_1,m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f\left(ta+(1-t)b, sc+m(1-s)d\right) \\
\leq\ & t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\
&+(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+m(1-s)d\right)dtds\right| \\
\leq\ & \int_0^1\int_0^1\left|t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
&+\int_0^1\int_0^1\left|(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{\alpha_1})^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p(1-s^{\alpha_1})^p dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}
\end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta} \leq 1-(1-t)^{\theta\beta}$ for $\theta > 0, \beta > 0$, we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{p\alpha_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})s^{p\alpha_1}dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})(1-s^{p\alpha_1})dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Theorem 3.6. *Let $f : \Delta = [0,b]\times[0,d] \to R$ be partial differentiable mapping on $\Delta = [0,b]\times[0,d]$ in R^2 with $0<a<b<\infty, 0<c<md<\infty, f \in L(\Delta), (\alpha_1, m) \in [0,1]^2$ and $\alpha \in R$. If $\left|f\right|$ is exponentially (α_1, m)-convex on the co-ordinates on Δ, $p,q>1, \frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds;*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \left(\frac{1}{p(p\alpha_1+1)^2}\right)+\frac{\left|f(a,c)\right|^q}{qe^{\alpha q(a+c)}} \\
&+\left(\frac{\alpha_1}{(p\alpha_1+1)^2}\right)+\left(\frac{\left|mf(a,d)\right|^q}{qe^{\alpha q(a+d)}}+\frac{\left|f(b,c)\right|^q}{qe^{\alpha q(b+c)}}\right) \\
&+\left(\frac{p\alpha_1^2}{(p\alpha_1+1)^2}\right)+\frac{\left|mf(b,d)\right|^q}{qe^{\alpha q(b+d)}}.
\end{aligned}$$

Proof. By the definition of the exponentially (α_1, m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f(ta+(1-t)b, sc+m(1-s)d) \\
\leq\ & t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\
&+(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}.
\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}
&\left|\int_0^1\int_0^1 f(ta+(1-t)b, sc+m(1-s)d)\,dtds\right| \\
\leq\ & \int_0^1\int_0^1\left|t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds \\
&+\int_0^1\int_0^1\left|(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds
\end{aligned}$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{\alpha_1})^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{\alpha_1})^p(1-s^{\alpha_1})^p dtds\right) \\
&+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)
\end{aligned}$$

By using the fact that $\left|1-(1-t)^{\theta}\right|^{\beta}\leq 1-(1-t)^{\theta\beta}$ for $\theta>0,\beta>0$, we can write

$$\begin{aligned}
&\left|\frac{1}{(b-a)(md-c)}\int_a^b\int_c^{md} f(x,y)dxdy\right| \\
\leq\ & \frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 t^{p\alpha_1}(1-s^{p\alpha_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})s^{p\alpha_1}dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right) \\
&+\frac{1}{p}\left(\int_0^1\int_0^1 (1-t^{p\alpha_1})(1-s^{p\alpha_1})dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{mf(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).
\end{aligned}$$

By computing the above integrals, we obtain the desired result. □

Proposition 3.7. *If $f,g:\Delta\to R$ are two exponentially (α_1,m)-convex on the co-ordinates on Δ, then $f+g$ is exponentially convex functions on the co-ordinates on Δ.*

Proof. By the definition of the exponentially (α_1,m)-convex on the co-ordinates on Δ, we can write

$$\begin{aligned}
& f\left(ta+(1-t)b,sc+m(1-s)d\right) \\
&+g\left(ta+(1-t)b,sc+m(1-s)d\right) \\
\leq\ & t^{\alpha_1}s^{\alpha_1}\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right) \\
&+t^{\alpha_1}(1-s^{\alpha_1})m\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right) \\
&+(1-t^{\alpha_1})s^{\alpha_1}\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right) \\
&+(1-t^{\alpha_1})(1-s^{\alpha_1})m\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right).
\end{aligned}$$

Namely,

$$\begin{aligned} & (f+g)\,(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{(f+g)\,(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{(f+g)\,(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{(f+g)\,(b,c)}{e^{\alpha(b+c)}} \\ & +(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{(f+g)\,(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore $(f+g)$ is exponentially (α_1, m)-convex on the co-ordinates on Δ. □

Proposition 3.8. *If $f : \Delta \to R$ is exponentially (α_1, m)-convex on the co-ordinates on Δ and $k \geq 0$ then kf is exponentially (α_1, m)-convex on the co-ordinates on Δ.*

Proof. By the definition of the exponentially (α_1, m)-convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned} & f\,(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{f(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{f(a,d)}{e^{\alpha(a+d)}} \\ & +(1-t^{\alpha_1})s^{\alpha_1}\frac{f(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m)\frac{f(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

If both sides are multiplied by k, we have,

$$\begin{aligned} & (kf)\,(ta+(1-t)b, sc+m(1-s)d) \\ \leq \quad & t^{\alpha_1}s^{\alpha_1}\frac{(kf)\,(a,c)}{e^{\alpha(a+c)}}+t^{\alpha_1}(1-s^{\alpha_1})m\frac{(kf)\,(a,d)}{e^{\alpha(a+d)}}+ \\ & (1-t^{\alpha_1})s^{\alpha_1}\frac{(kf)\,(b,c)}{e^{\alpha(b+c)}}+(1-t^{\alpha_1})(1-s^{\alpha_1})m\frac{(kf)\,(b,d)}{e^{\alpha(b+d)}}. \end{aligned}$$

Therefore (kf) is exponentially (α_1, m)-convex functions on the co-ordinates on Δ. □

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