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A Work on Cauchy Problems for Variable-Order Derivatives with Exponential Kernel

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Abstract. In this work, we study the existence and uniqueness of solutions to variable-order fractional differential equations with exponential kernel. The Caputo-Fabrizio definition is used for the variable-order fractional derivative with the order being a bounded function. To prove the existence of solutions for the Cauchy problem, we develop an iterative sequence. Finally, uniqueness is verified via linear growth and Lipschitz conditions.

1. Introduction

The exponential kernel is a significant approach for analyzing complex systems in fields such as financial markets, biological processes, and engineering sciences, particularly for capturing phenomena where changes occur rapidly over time. Exponential functions grow or decay at rates proportional to their current magnitude, making them highly responsive to rapid increases or decreases in value. When examining the relationship between exponential functions and variable-order derivatives, it becomes evident that exponential functions, characterized by growth or decay rates proportional to their initial values, serve as kernels in fractional calculus. When combined with variable-order derivatives, this approach facilitates the modeling of many systems where the order of differentiation changes over time or based on system properties. By using an exponential kernel, the model is particularly effective in representing processes with rapid changes, making it applicable to fields such as finance, biology, and engineering. Our results significantly extend many of the existing theorems on existence and uniqueness in the literature [1-4,9-10].

Let us present the Caputo-Fabrizio derivative definition and theorem, which is the main focus of our study as below [5].

Definition 1 :Let $f \in H^1(a,b)$, $b > a$, $0 < \alpha < 1$ then, the Caputo-Fabrizio derivative of fractional derivative is defined as :

$$ {}_a^{CF}D_t^{\alpha} f\left(t\right) = \frac{1}{1-\alpha}\int_a^t f^{'}\left(\tau\right)\exp\left[-\alpha\frac{\left(t-\tau\right)}{1-\alpha}\right]d\tau. \tag{1}$$

Theorem 1 :Let $0 < \alpha < 1$ then the following time fractional ordinary differential equation

$$ {}_0^{CF}D_t^{\alpha} y\left(t\right) = f(t, y(t)), \tag{2}$$

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has a unique solution by taking the inverse Laplace transform and applying the convolution theorem as follows:

$$y(t) = (1-\alpha) f(t, y(t)) + \alpha \int_0^t f(\tau, y(\tau)) d\tau, \ t \geq 0. \tag{3}$$

By substituting the fractional order in derivatives and integrals with a bounded function, we obtain variable-order fractional derivatives and integrals. These include both integer and fractional order cases as special instances. Similar results have been presented in [8].

2. Existence and Uniqueness of Solutions for Cauchy Problems Involving Variable-Order Fractional Derivatives with Exponential Kernel

In this section, we consider the Cauchy problem for variable-order derivative with exponential kernel. To show this let us consider the following general Caputo Fabrizio integral equation

$$y(t) = y_0 + (1-\alpha(t)) f(t, y(t)) + \alpha(t) \int_{t_0}^t f(\tau, y(\tau)) d\tau. \tag{4}$$

Here we want to note that this operator is not invertible with the Caputo-Fabrizio derivative wtih variable order.

To ensure applicability to models, the order function is usually limited to $0 < \alpha(t) \leq 1$. Moreover, in this work, we focus on $\alpha(t)$ in two intervals where

$$h_1 = \min\{\alpha(t), t \in I\}$$

and

$$h_2 = \max\{\alpha(t), t \in I\}$$

are the minimum and maximum value of the order function $\alpha(t)$ on the interval I, respectively. In this work also we defined the following norm

$$\left\|y\right\|_\infty = \sup_{t \in D_y} \left|y(t)\right|. \tag{5}$$

Let us give now sufficient conditions for which the Cauchy problems for variable-order fractional derivative with power kernel has unique equation if the nonlinear function $f(t, y(t))$ satisfies the following conditions [6-7].

1) $\forall t \in [0, T]$, the function $\alpha(t)$ is a differentiable and bounded nonzero and non-constant function and

$$0 < h_1 < \alpha(t) < h_2 \leq 1.$$

2) $\forall t \in [0, T]$, $\alpha'(t) \neq 0$.

3) $f(t, y(t))$ is a nonlinear function and twice differentiable and bounded.

4) $\left|f(t, y(t))\right|^2 < L\left(1 + \left|y\right|^2\right)$ (Linear growth condition).

5) $\left|f(t, y_1(t)) - f(t, y_2(t))\right|^2 < \overline{L}\left|y_1 - y_2\right|^2$ (Lipschitz condition).

Then the Cauchy problem has a unique solution in $L^2([t_0, T], R)$. Assume that the linear growth conditon satisfy then, $\forall_l \geq 1$, We can apply the stopping duration as below

$$\lambda_l = \inf\{T, \ \inf\left\{t \in [t_0, T] : \left|y(t)\right| > l\right\}\}. \tag{6}$$

So we have that $\lim_{l \to \infty} \lambda_l = T$. Then we can define the sequence

$$y_l(t) = y(\inf(t, \lambda_l)), \ \forall_t \in [t_0, T]. \tag{7}$$

$\forall_l \geq 1$, we defined the following sequence

$$y_l(t) = y_0 + (1 - \alpha(t)) f(t, y_l(t)) + \alpha(t) \int_{t_0}^{t} f(\tau, y_l(\tau)) d\tau. \tag{8}$$

Then

$$\left|y_l(t)\right|^2 \leq 3\left|y_0\right|^2 + 3\left|(1 - h_2) f(t, y_l(t))\right|^2 + 3\left|\int_{t_0}^{t} h_2 f(\tau, y_l(\tau)) d\tau\right|^2, \tag{9}$$

then using the linear growth hypothesis gives

$$\begin{aligned} \left|y_l(t)\right|^2 &\leq 3\left|y_0\right|^2 + 3L\left(1 + \left|y_l(t)\right|^2\right)(1 - h_2)^2 \\ &\quad + 3Lh_2^2 \int_{t_0}^{t} \left(1 + \left|y_l(\tau)\right|^2\right) d\tau, \end{aligned} \tag{10}$$

$$\begin{aligned} \max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2 &\leq 3\left|y_0\right|^2 + 3L\left(1 + \max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right)(1 - h_2)^2 \\ &\quad + 3Lh_2^2 \int_{t_0}^{t} \left(1 + \max_{t_0 \leq r \leq \tau} \left|y_l(r)\right|^2\right) d\tau. \end{aligned} \tag{11}$$

With the help of expectation, we determine

$$\begin{aligned} E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right) &\leq 3\left|y_0\right|^2 + 3L\left(1 + E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right)\right)(1 - h_2)^2 \\ &\quad + 3Lh_2^2 \int_{t_0}^{t} \left(1 + E\left(\max_{t_0 \leq r \leq \tau} \left|y_l(r)\right|^2\right)\right) d\tau. \end{aligned} \tag{12}$$

After adding 1 to both sides, we get

$$\begin{aligned} 1 + E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right) &\leq 1 + 3\left|y_0\right|^2 + 3L\left(1 + E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right)\right)(1 - h_2)^2 \\ &\quad + 3Lh_2^2 \int_{t_0}^{t} \left(1 + E\left(\max_{t_0 \leq r \leq \tau} \left|y_l(r)\right|^2\right)\right) d\tau. \end{aligned} \tag{13}$$

So finally we get

$$\begin{aligned} E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right) &\leq 1 + 3\left|y_0\right|^2 + 3L\left(1 + E\left(\max_{t_0 \leq k \leq t} \left|y_l(k)\right|^2\right)\right)(1 - h_2)^2 \\ &\quad + \exp\left[3Lh_2^2 (T - t_0)\right]. \end{aligned} \tag{14}$$

Let $y_1(t)$ and $y_2(t)$ be two solutions of the problem. Then $y_1(t), y_2(t) \in L^2([t_0, T], R)$. So we have

$$\begin{aligned} \left|y_1(t) - y_2(t)\right|^2 &\leq 2\left|(1 - h_2)(f(t, y_1(t)) - f(t, y_2(t)))\right|^2 \\ &\quad + 2\left|\int_{t_0}^{t} h_2 (f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))) d\tau\right|^2, \end{aligned} \tag{15}$$

using the Lipschitz conditon for funtion $f(t, y(t))$ then we get

$$\left|y_1(t) - y_2(t)\right|^2 \leq 2\overline{L}(1-h_2)^2\left|y_1(t) - y_2(t)\right|^2 + 2\overline{L}h_2^2\int_{t_0}^{t}\left|y_1(\tau) - y_2(\tau)\right|^2 d\tau, \tag{16}$$

thus

$$E\left(\sup_{t_0\leq k\leq t}\left|y_1(k) - y_2(k)\right|^2\right) \leq 2\overline{L}(1-h_2)^2 E\left(\sup_{t_0\leq k\leq t}\left|y_1(k) - y_2(k)\right|^2\right) + 2\overline{L}h_2^2\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq \tau}\left|(y_1(r) - y_2(r))\right|^2\right)d\tau, \tag{17}$$

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) \leq 2\overline{L}(1-h_2)^2 E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) + 2\overline{L}h_2^2\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq \tau}\left|(y_1(r) - y_2(r))\right|^2\right)d\tau, \tag{18}$$

$$\left(1 - 2\overline{L}(1-h_2)^2\right)E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) \leq 2\overline{L}h_2^2\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq \tau}\left|(y_1(r) - y_2(r))\right|^2\right)d\tau, \tag{19}$$

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) \leq \frac{2\overline{L}h_2^2\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq \tau}\left|(y_1(r) - y_2(r))\right|^2\right)d\tau}{\left(1 - 2\overline{L}(1-h_2)^2\right)}, \tag{20}$$

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) \leq \overline{f}\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq \tau}\left|(y_1(r) - y_2(r))\right|^2\right)d\tau, \tag{21}$$

where

$$\overline{f} = \frac{2\overline{L}h_2^2}{\left(1 - 2\overline{L}(1-h_2)^2\right)}, \tag{22}$$

with following condition

$$2\overline{L}(1-h_2)^2 \neq 1. \tag{23}$$

Applying Gronwall inequality

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t) - y_2(t)\right|^2\right) = 0, \forall t \in [t_0, T]. \tag{24}$$

So we have

$$y_1(t) = y_2(t)\ \forall t \in [t_0, T]. \tag{25}$$

2.1. Existence of solution

In this subsection, we present a proof of the existence of the solution by utilizing the Picard iterative method. The Picard iterative approach is a powerful tool in the analysis of differential equations, particularly for proving the existence and uniqueness of solutions. By constructing a sequence of approximations that converge to the true solution, this method leverages successive iterations to refine the approximation at each step. Considering the recursive formula on l, we get following equality for solution of Cauchy problem

$$y_l(t) = (1-\alpha(t))\, f(t, y_{l-1}(t)) + \alpha(t)\int_{t_0}^{t} f(\tau, y_{l-1}(\tau))d\tau. \tag{26}$$

Now we have to show that $\forall_l \geq 0$, $y_l(t) \in L^2([t_0, T], R)$. $y_0(t) = y_0$ is the initial condition by definition $y_l(t) \in L^2([t_0, T], R)$. For $l = 1$,

$$\begin{aligned} y_1(t) &= (1-\alpha(t))\, f(t, y_0(t)) + \alpha(t)\int_{t_0}^{t} f(\tau, y_0(\tau))d\tau, \\ &\leq (1-h_2)\, f(t, y_0(t)) + h_2\int_{t_0}^{t} f(\tau, y_0(\tau))d\tau. \end{aligned} \tag{27}$$

Consequently, we obtain

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t)\right|^2\right) \leq 2L\,(1-h_2)^2\left(1+E\left|y_0\right|^2\right) + 2Lh_2^2\left(1+E\left|y_0\right|^2\right)(T-t_0)\,, \tag{28}$$

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t)\right|^2\right) \leq \left(2L\left(1+E\left|y_0\right|^2\right)\right)\left((1-h_2)^2 + h_2^2\,(T-t_0)\right). \tag{29}$$

We know that $y_0 \in L^2([t_0, T], R)$, thus $E\left|y_0\right|^2 < \infty$. In that case

$$E\left(\sup_{t_0\leq t\leq T}\left|y_1(t)\right|^2\right) < \infty. \tag{30}$$

We assume that $\forall_l \geq 1$, $y_l(t) \in L^2([t_0, T], R)$. At this point, it is necessary to demonstrate that $y_{l+1}(t) \in L^2([t_0, T], R)$,

$$\begin{aligned} y_{l+1}(t) &= (1-\alpha(t))\, f(t, y_l(t)) + \alpha(t)\int_{t_0}^{t} f(\tau, y_l(\tau))d\tau, \\ &\leq (1-h_2)\, f(t, y_l(t)) + h_2\int_{t_0}^{t} f(\tau, y_l(\tau))d\tau. \end{aligned} \tag{31}$$

So we get

$$\left|y_{l+1}(t)\right|^2 \leq 2L\,(1-h_2)^2\left(1+\left|y_l(t)\right|^2\right) + 2Lh_2^2\int_{t_0}^{t}\left(1+\left|y_l(\tau)\right|^2\right)d\tau, \tag{32}$$

$$E\left(\sup_{t_0\le t\le T}\left|y_{l+1}(t)\right|^2\right) \le 2L(1-h_2)^2\left(1+E\left(\sup_{t_0\le t\le T}\left|y_l(t)\right|^2\right)\right) +2Lh_2^2\left(1+E\left(\sup_{t_0\le r\le \tau}\left|y_l(r)\right|^2\right)\right)d\tau. \tag{33}$$

Based on the inductive hypothesis, we get $y_l(t)\in L^2([t_0,T],R)$ then we have

$$E\left(\sup_{t_0\le t\le T}\left|y_l(t)\right|^2\right)\le \Phi. \tag{34}$$

So we obtain

$$E\left(\sup_{t_0\le t\le T}\left|y_{l+1}(t)\right|^2\right) \le 2L(1+\Phi)\left\{(1-h_2)^2+h_2^2(T-t_0)\right\}, < \infty. \tag{35}$$

Therefore $y_{l+1}(t)\in L^2([t_0,T],R)$. Using inductive principles, we can state that $\forall_l\ge 0$, $y_l(t)\in L^2([t_0,T],R)$. We now analyze,

$$E\left(\left|y_1(t)-y_0\right|^2\right) \le 3L(1-h_2)^2\left(1+E\left(\left|y_0\right|^2\right)\right)+3E\left(\left|y_0\right|^2\right) +3(T-t_0)L\left(1+E\left(\left|y_0\right|^2\right)\right)h_2^2, \tag{36}$$

and

$$E\left(\left|y_1(t)-y_0\right|^2\right)\le\gamma_1, \tag{37}$$

where

$$\gamma_1 = 3L(1-h_2)^2\left(1+E\left(\left|y_0\right|^2\right)\right)+3E\left(\left|y_0\right|^2\right) +3(T-t_0)L\left(1+E\left(\left|y_0\right|^2\right)\right)h_2^2. \tag{38}$$

Now $\forall_l\ge 1$,

$$E\left(\sup_{t_0\le t\le T}\left|y_1(t)\right|^2\right)\le\left(2L\left(1+E\left|y_0\right|^2\right)\right)\left\{(1-h_2)^2+h_2^2(T-t_0)\right\}, \tag{39}$$

$$\left|y_{l+1}(t)-y_l(t)\right|^2 \le 2\overline{L}(1-h_2)^2\left|y_l(t)-y_{l-1}(t)\right|^2 +2\overline{L}h_2^2\int_{t_0}^{t}\left|y_l(\tau)-y_{l-1}(\tau)\right|^2d\tau. \tag{40}$$

Considering induction for $\forall_l\ge 0$, we get

$$E\left(\sup_{t_0\le t\le T}\left|y_{l+1}(t)-y_l(t)\right|^2\right)\le\gamma_1\frac{(\beta_1(t-t_0))^l}{l!},\ t_0\le t\le T. \tag{41}$$

We take the inequality to be valid for every $l\ge 1$. Now we must show its prove at $l+1$. At $l+1$, we have

$$E\left(\sup_{t_0\le t\le T}\left|y_{l+2}(t)-y_{l+1}(t)\right|^2\right)\le\gamma_1\frac{(\beta_1(t-t_0))^{l+1}}{(l+1)!},\ t_0\le t\le T. \tag{42}$$

So at $l+1$, the validity of the inequality is established using the inductive principle.

We can conclude that the Borel-Cantelli lemma helps to find a positive integer number $l_0 = l_0(\varepsilon)$, $\forall \varepsilon \in \Pi$ that

$$\sup_{t_0 \le t \le T} \left| y_{l+1}(t) - y_l(t) \right|^2 \le \frac{1}{2^l},\ l \ge l_0. \tag{43}$$

It continues that the sum

$$y_0(t) + \sum_{k=0}^{l-1} \left[y_{k+1}(t) - y_k(t) \right] = y_l(t), \tag{44}$$

converges uniformly in $[0, T]$. Now, if we take

$$\lim_{l \to \infty} y_l(t) = y(t). \tag{45}$$

As a result, we have

$$E \left| y_{l+1}(t) - y(t) \right|^2 \le \beta_1 \left| y_l(t) - y(t) \right|^2. \tag{46}$$

Taking as $l \to \infty$, the right side of equality goes to zero, so we obtain

$$y(t) = (1 - \alpha(t))\, f(t, y(t)) + \alpha(t) \int_{t_0}^{t} f(\tau, y(\tau)) d\tau. \tag{47}$$

This completes the proof.

3. Conlusion

In this paper, we have investigated the Cauchy problem for differential equations with variable-order derivatives using the Caputo-Fabrizio fractional derivative operator, highlighting the importance of these advanced mathematical tools in modeling complex real-world phenomena. The investigation into the existence and uniqueness of solutions has demonstrated that variable-order derivatives, combined with the Caputo-Fabrizio operator provide an effective framework for capturing memory effects in traditional fractional derivative models.

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