TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 3, 241–251
ISSN: 2587–0971

https://www.tjoscience.com

Integral Inequalities Involving Modified Exponential Trigonometric Convex Functions

Mustafa GÜRBÜZ[a]

[a]*Department of Primary Mathematics Education, Faculty of Education, University of Ağrı İbrahim Çeçen, Ağrı, TURKEY.*

Abstract. In this study, inequalities have been derived for the class of modified exponential trigonometric convex function, a type of convexity recently introduced. To establish these inequalities, the Hölder inequality, the power mean inequality, and the generalized power mean inequality have been utilized.

1. Introduction

Inequalities are increasingly attracting the attention of researchers across diverse fields of application. Convexity, which extends back to the proof of the renowned π number, plays a significant role in establishing inequalities. In this section, the definitions of fundamental classes of convex functions are first presented, followed by the introduction of a new function class proposed by Demir Budak and Gürbüz [6]. Finally, the improved power-mean integral inequality and a lemma which will be very useful to obtain our results are presented. The definition of a convex function is as follows:

Definition 1.1. *A function $f : I \to \mathbb{R}$ is said to be convex if the inequality*

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1}$$

is valid for all $a, b \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics.

Let $f : I \to \mathbb{R}$ be a convex function. Then the following double inequality hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave.

Corresponding author: MG: mgurbuz@agri.edu.tr ORCID:0000-0002-7092-4298
Received: 22 October 2024; Accepted: 26 November 2024; Published: 31 December 2024.
Keywords. Modified exponential trigonometric convexity, integral inequalities.
2010 *Mathematics Subject Classification*. 26D10, 26A51
Cited this article as: Gürbüz, M., (2024). Integral inequalities involving modified exponential trigonometric convex functions. Turkish Journal of Science, 9(3), 241–251.

Definition 1.2. *(see [4]) Suppose that I is a subset of* $\mathbb{R}$.*The mapping* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be exponentially convex, if*

$$f(ta + (1-t)b) \leq te^{xa} f(a) + (1-t)e^{xb} f(b) \tag{3}$$

for all $a, b \in I, t \in [0,1]$ *and* $x \in \mathbb{R}$.

Definition 1.3. *(see [1]) A non-negative function* $f : I \to \mathbb{R}$ *is called trigonometrically convex function on interval* $[a, b]$, *if for each* $x, y \in [a, b]$ *and* $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \left(\frac{\sin \pi t}{2}\right) f(x) + \left(\frac{\cos \pi t}{2}\right) f(y). \tag{4}$$

Definition 1.4. *(see [5]) Let* $I \subseteq \mathbb{R}$ *be an interval. A function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is called exponential trigonometric convex function if for every* $a, b \in I$, $t \in [0, 1]$

$$f(ta + (1-t)b) \leq \frac{\sin \frac{\pi t}{2}}{e^{1-t}} f(a) + \frac{\cos \frac{\pi t}{2}}{e^{t}} f(b). \tag{5}$$

Definition 1.5. *(see [6])* $f : I \subseteq \mathbb{R} \to \mathbb{R}_0^+$ *is called modified exponential trigonometric convex function if for every* $a, b \in I$ *and* $t \in [0, 1]$ *the following inequality holds*

$$f(ta + (1-t)b) \leq \sin \frac{\pi t}{2} e^{(1-t)} f(a) + \cos \frac{\pi t}{2} e^{t} f(b). \tag{6}$$

This class of functions defined on I will be shown by METC(I).

A refinement of power-mean integral inequality can be given as follows:

Theorem 1.6. *(see [2]) Let* $q \geq 1$. *If f and g are real functions defined on interval* $[a, b]$ *and if* $|f|$, $|f|\,|g|^q$ *are integrable functions on* $[a, b]$ *then*

$$\int_a^b |f(x)\, g(x)|\, dx \tag{7}$$

$$\leq \frac{1}{b-a} \left\{ \begin{array}{c} \left(\int_a^b (b-x)\,|f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}} \\ + \left(\int_a^b (x-a)\,|f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}} \end{array} \right\}.$$

Improved power-mean integral inequality provides better approaches than Power-mean integral inequality.

Lemma 1.7. *(see [3])Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$. *If* $f' \in L[a, b]$, *then the following equality holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \tag{8}$$

$$= \frac{b-a}{4} \left[\int_0^1 (-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt + \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) dt \right].$$

Throughout the paper we will denote $\mathbb{R}^+ \cup \{0\}$ *by* $\mathbb{R}_0^+$.

There are many papers that present upper bounds for Hermite-Hadamard type, specifically trapezoid type inequalities. To obtain those results, researchers are using many different kinds of convex functions. One of the most recent definitions of convex function types is "modified exponential trigonometric convex function" which serves another viewpoint to convexity. The motivation of this study is to obtain new upper bounds of Hermite-Hadamard type inequalities by using modified exponential trigonometric convex function which may be very useful for applications of inequalities. Researchers interested in exploring recent intriguing studies on inequalities can refer to references [7]-[11].

2. MAIN RESULTS

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_0^+$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $|f'|$ is a modified exponential trigonometric convex function on interval $[a,b]$, then the following inequaltiy holds*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\ \leq\ & \frac{8(b-a)\left(|f'(a)| + |f'(b)|\right)}{(\pi^2+4)^2}\left(\frac{\pi^3 e}{2} + \pi^2 - 12 - 6\pi e + \sqrt{2e}\left(8 - 2\pi^2 + 8\pi\right)\right). \end{aligned}$$

Proof. By using Lemma 1.7, properties of absolute value and modified exponential trigonometric convexity of $|f'|$ we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \qquad (9)\\ \leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|dt + \int_0^1 t\left|f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right|dt\right] \\ \leq\ & \frac{b-a}{4}\left[\int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(a)\right| + \cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(b)\right|\right)dt\right. \\ & \left. + \int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(b)\right| + \cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(a)\right|\right)dt\right] \\ =\ & \frac{b-a}{4}\left[\left|f'(a)\right|\int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}} + \cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\right)dt\right. \\ & \left. + \left|f'(b)\right|\int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}} + \cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\right)dt\right] \\ =\ & \frac{8(b-a)\left(|f'(a)| + |f'(b)|\right)}{(\pi^2+4)^2}\left(\frac{\pi^3 e}{2} + \pi^2 - 12 - 6\pi e + \sqrt{2e}\left(8 - 2\pi^2 + 8\pi\right)\right) \end{aligned}$$

where

$$\begin{aligned}&\int_0^1 t\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}dt\\ =\ &\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}+\pi^2+4\sqrt{2}\pi e^{\frac{1}{2}}-\sqrt{2}\pi^2 e^{\frac{1}{2}}-12\right)\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}dt\\ =\ &\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}-\sqrt{2}\pi^2 e^{\frac{1}{2}}+4\sqrt{2}\pi e^{\frac{1}{2}}-6\pi e+\frac{\pi^3 e}{2}\right)\end{aligned}$$

which completes the proof. □

Theorem 2.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}_0^+$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and assume that $f'\in L[a,b]$. If $|f'|^q$, $q>1$ is a modified exponential trigonometric convex function on interval $[a,b]$, then the following inequaltiy holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (10)\\ \le\ &\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(|f'(a)|^q\left(\frac{\sqrt{2e}(4+2\pi)-8}{\pi^2+4}\right)-|f'(b)|^q\left(\frac{\sqrt{2e}(4+2\pi)-4\pi e}{\pi^2+4}\right)\right)^{\frac{1}{q}}\right.\\ &\left.+\left(|f'(b)|^q\left(\frac{\sqrt{2e}(4+2\pi)-8}{\pi^2+4}\right)-|f'(a)|^q\left(\frac{\sqrt{2e}(4+2\pi)-4\pi e}{\pi^2+4}\right)\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 1.7, properties of absolute value and Hölder inequality we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (11)\\ \le\ &\frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]\\ \le\ &\frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Using modified exponential trigonometric convexity of $\left|f'\right|$ we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\le\ &\frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(a)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(b)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
=\ &\frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q\left(\frac{\sqrt{2e}(4+2\pi)-8}{\pi^2+4}\right)-\left|f'(b)\right|^q\left(\frac{\sqrt{2e}(4+2\pi)-4\pi e}{\pi^2+4}\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left|f'(b)\right|^q\left(\frac{\sqrt{2e}(4+2\pi)-8}{\pi^2+4}\right)-\left|f'(a)\right|^q\left(\frac{\sqrt{2e}(4+2\pi)-4\pi e}{\pi^2+4}\right)\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where

$$\int_0^1 \sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}dt=\frac{\sqrt{2e}(4+2\pi)-8}{\pi^2+4}$$

and

$$\int_0^1 \cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}dt=-\frac{\sqrt{2e}(4+2\pi)-4\pi e}{\pi^2+4}.$$

This completes the proof of theorem. □

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}_0^+$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$ and assume that $f' \in L[a,b]$. If $\left|f'\right|^q$, $q \ge 1$ is a modified exponential trigonometric convex function on interval $[a,b]$, then the following inequaltiy holds*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \qquad (12)\\
\le\ &\frac{b-a}{2^{3-\frac{4}{q}}(\pi^2+4)^{\frac{2}{q}}}\left[\left(K\left|f'(a)\right|^q+L\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(K\left|f'(b)\right|^q+L\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where

$$K=\sqrt{2e}\left(4+4\pi-\pi^2\right)+\pi^2-12$$

and

$$L=\sqrt{2e}\left(4+4\pi-\pi^2\right)-6\pi e+\frac{\pi^3 e}{2}.$$

Proof. By using Lemma 1.7, properties of absolute value and power mean inequality we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\;& \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right] \\
\leq\;& \frac{b-a}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{13}$$

Using modified exponential trigonometric convexity of $|f'|$ we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\;& \frac{b-a}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(a)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(b)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
=\;& \frac{b-a}{2^{3-\frac{1}{q}}}\left[\left(\left|f'(a)\right|^q\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}+\pi^2+4\sqrt{2}\pi e^{\frac{1}{2}}-\sqrt{2}\pi^2e^{\frac{1}{2}}-12\right)\right.\right. \\
&\left.+\left|f'(b)\right|^q\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}-\sqrt{2}\pi^2e^{\frac{1}{2}}+4\sqrt{2}\pi e^{\frac{1}{2}}-6\pi e+\frac{\pi^3e}{2}\right)\right)^{\frac{1}{q}} \\
&\left(+\left|f'(b)\right|^q\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}+\pi^2+4\sqrt{2}\pi e^{\frac{1}{2}}-\sqrt{2}\pi^2e^{\frac{1}{2}}-12\right)\right. \\
&\left.\left.+\left|f'(a)\right|^q\frac{8}{(\pi^2+4)^2}\left(4\sqrt{2}e^{\frac{1}{2}}-\sqrt{2}\pi^2e^{\frac{1}{2}}+4\sqrt{2}\pi e^{\frac{1}{2}}-6\pi e+\frac{\pi^3e}{2}\right)\right)^{\frac{1}{q}}\right]
\end{aligned}$$

which is the desired result. □

Theorem 2.4. *Let the function $f: I\subset\mathbb{R}\to\mathbb{R}_0^+$ be differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, $q>1$ and assume that $f'\in L[a,b]$. If $\left|f'\right|^q$ is a modified exponential trigonometric convex function on the interval $[a,b]$, then* *the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{14}$$
$$\leq \frac{(b-a)\,2^{\frac{7}{2q}-2}e^{\frac{1}{2q}}}{(\pi^2+4)^{\frac{3}{q}}}$$
$$\times\Bigg[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(\Bigg(-48+48\pi^2+12\pi^3-\pi^4-\frac{48\pi^2\sqrt{2}}{\sqrt{e}}-\sqrt{2}\pi^4-80\pi+80\sqrt{2}\Bigg)\left|f'(a)\right|^q$$
$$+\left(-48\pi^2+80-4\pi^3-\pi^4+112\pi-80\pi\sqrt{2e}+12\pi^3\sqrt{2e}\right)\left|f'(b)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg(\frac{1}{2}\Bigg(128-96\pi^2-16\pi^3+\frac{88\pi^2\sqrt{2}}{\sqrt{e}}+\frac{3\pi^4\sqrt{2}}{\sqrt{e}}+192\pi-\frac{208\sqrt{2}}{\sqrt{e}}\Bigg)\left|f'(a)\right|^q$$
$$+\frac{1}{4}\left(192\pi^2-256+32\pi^3-384\pi+272\pi\sqrt{2e}-56\pi^3\sqrt{2e}+\pi^5\sqrt{2e}\right)\left|f'(b)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(\Bigg(-48+48\pi^2+12\pi^3-\pi^4-\frac{48\pi^2\sqrt{2}}{\sqrt{e}}-\frac{\pi^4\sqrt{2}}{\sqrt{e}}-80\pi+\frac{80\sqrt{2}}{\sqrt{e}}\Bigg)\left|f'(b)\right|^q$$
$$\left(-48\pi^2+80-4\pi^3-\pi^4+112\pi-80\pi\sqrt{2e}+12\pi^3\sqrt{2e}\right)\left|f'(a)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\Bigg(\frac{1}{2}\Bigg(128-96\pi^2-16\pi^3+\frac{88\pi^2\sqrt{2}}{\sqrt{e}}+\frac{3\pi^4\sqrt{2}}{\sqrt{e}}+192\pi-\frac{208\sqrt{2}}{\sqrt{e}}\Bigg)\left|f'(b)\right|^q$$
$$+\frac{1}{4}\left(192\pi^2-256+32\pi^3-384\pi+272\pi\sqrt{2e}-56\pi^3\sqrt{2e}+\pi^5\sqrt{2e}\right)\left|f'(a)\right|^q\Bigg)^{\frac{1}{q}}\Bigg].$$

Proof. By using Lemma 1.7, properties of absolute value and improved power-mean inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{15}$$
$$\leq \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\int_0^1 t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right]$$
$$\leq \frac{b-a}{4}\left[\left(\int_0^1 (1-t)\,t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \tag{16}$$

$$+\left(\int_0^1 t^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-t)\,tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^2 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

which can be rewritten as

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{b-a}{4}\Bigg[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

Since $\left|f'\right|$ is a modified exponential trigonometric convex function we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{b-a}{4}\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(a)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(a)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(b)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left(\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}\left|f'(b)\right|^q+\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

By simple calculation

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{b-a}{4}\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\sqrt{e}-48\pi^2\sqrt{e}-12\pi^3\sqrt{e}\right.\right.\right. \\
&\left.+\pi^4\sqrt{e}+48\sqrt{2}\pi^2+\sqrt{2}\pi^4+80\pi\sqrt{e}-80\sqrt{2}\right)\left|f'(a)\right|^q \\
&+\left(\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\pi^2\sqrt{e}-80\sqrt{e}+4\pi^3\sqrt{e}\right.\right. \\
&\left.\left.+\pi^4\sqrt{e}-112\pi\sqrt{e}+80\sqrt{2}\pi e-12\sqrt{2}\pi^3 e\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{4\sqrt{2}}{(\pi^2+4)^3}\left(128\sqrt{e}-96\pi^2\sqrt{e}-16\pi^3\sqrt{e}\right.\right. \\
&\left.+88\sqrt{2}\pi^2+3\sqrt{2}\pi^4+192\pi\sqrt{e}-208\sqrt{2}\right)\left|f'(a)\right|^q \\
&+\left(\frac{2\sqrt{2}}{(\pi^2+4)^3}\left(192\pi^2\sqrt{e}-256\sqrt{e}+32\pi^3\sqrt{e}\right.\right. \\
&\left.\left.-384\pi\sqrt{e}+272\sqrt{2}\pi e-56\sqrt{2}\pi^3 e+\sqrt{2}\pi^5 e\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\sqrt{e}-48\pi^2\sqrt{e}-12\pi^3\sqrt{e}\right.\right.\\
&\left.+\pi^4\sqrt{e}+48\sqrt{2}\pi^2+\sqrt{2}\pi^4+80\pi\sqrt{e}-80\sqrt{2}\right)\left|f'(b)\right|^q\\
&+\left(\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\pi^2\sqrt{e}-80\sqrt{e}+4\pi^3\sqrt{e}\right.\right.\\
&\left.\left.+\pi^4\sqrt{e}-112\pi\sqrt{e}+80\sqrt{2}\pi e-12\sqrt{2}\pi^3 e\right)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{4\sqrt{2}}{(\pi^2+4)^3}\left(128\sqrt{e}-96\pi^2\sqrt{e}-16\pi^3\sqrt{e}\right.\right.\\
&\left.+88\sqrt{2}\pi^2+3\sqrt{2}\pi^4+192\pi\sqrt{e}-208\sqrt{2}\right)\left|f'(b)\right|^q\\
&+\left(\frac{2\sqrt{2}}{(\pi^2+4)^3}\left(192\pi^2\sqrt{e}-256\sqrt{e}+32\pi^3\sqrt{e}\right.\right.\\
&\left.\left.\left.-384\pi\sqrt{e}+272\sqrt{2}\pi e-56\sqrt{2}\pi^3 e+\sqrt{2}\pi^5 e\right)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_0^1 (1-t)\,t\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}dt\\
=\;&\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\sqrt{e}-48\pi^2\sqrt{e}-12\pi^3\sqrt{e}\right.\\
&\left.+\pi^4\sqrt{e}+48\sqrt{2}\pi^2+\sqrt{2}\pi^4+80\pi\sqrt{e}-80\sqrt{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 (1-t)\,t\cos\frac{\pi(1+t)}{4}e^{\frac{1+t}{2}}dt\\
=\;&\frac{-8\sqrt{2}}{(\pi^2+4)^3}\left(48\pi^2\sqrt{e}-80\sqrt{e}+4\pi^3\sqrt{e}\right.\\
&\left.+\pi^4\sqrt{e}-112\pi\sqrt{e}+80\sqrt{2}\pi e-12\sqrt{2}\pi^3 e\right)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 t^2\sin\frac{\pi(1+t)}{4}e^{\frac{1-t}{2}}dt\\
=\;&\frac{4\sqrt{2}}{(\pi^2+4)^3}\left(128\sqrt{e}-96\pi^2\sqrt{e}-16\pi^3\sqrt{e}\right.\\
&\left.+88\sqrt{2}\pi^2+3\sqrt{2}\pi^4+192\pi\sqrt{e}-208\sqrt{2}\right)
\end{aligned}
$$

and

$$\int_0^1 t^2 \cos\frac{\pi(1+t)}{4} e^{\frac{1+t}{2}} dt$$
$$= \frac{2\sqrt{2}}{(\pi^2+4)^3}\Big(192\pi^2\sqrt{e} - 256\sqrt{e} + 32\pi^3\sqrt{e}$$
$$-384\pi\sqrt{e} + 272\sqrt{2}\pi e - 56\sqrt{2}\pi^3 e + \sqrt{2}\pi^5 e\Big)$$

which is the desired result. □

References

[1] Kadakal, H. "Hermite-Hadamard type inequalities for trigonometrically convex functions." Sci. Stud. Res. Ser. Math. Inform 28(2) (2018): 19-28.
[2] Kadakal, M; İşcan, İ; Kadakal, H; Bekar, K. "On improvements of some integral inequalities." Honam Math. J. 43 (2021), No. 3, 441–452.
[3] Alomari, M., Darus, M., & Kirmaci, U. S. Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means. Computers & mathematics with applications, 59(1), 225-232 (2010).
[4] Awan, M.U.; Noor, M.A.; Noor, K.I. Hermite-Hadamard inequalities for exponentiaaly convex functions. Appl. Math. Inform Sci., 2018, 2, 405–409.
[5] Kadakal, M., İşcan, İ., Agarwal, P., & Jleli, M. (2021). Exponential trigonometric convex functions and Hermite-Hadamard type inequalities. Mathematica Slovaca, 71(1), 43-56.
[6] Budak Demir, K., Gürbüz, M. Modified exponential trigonometric convex functions and some integral inequalities. Turkish J. Ineq, 8(2), 60-72 (2024).
[7] Lakhdari, A., Budak, H., Awan, M. U., & Meftah, B. Extension of Milne-type inequalities to Katugampola fractional integrals. Boundary Value Problems, 2024(1), 100 (2024).
[8] Ying, R., Lakhdari, A., Xu, H., Saleh, W., & Meftah, B. On Conformable Fractional Milne-Type Inequalities. Symmetry, 16(2), 196 (2024).
[9] Samraiz, M., Atta, T., Naheed, S., Abdeljawad, T., & Ghaffar, M. T. A novel class of integral inequalities with graphical approach and diverse applications. Mathematical and Computer Modelling of Dynamical Systems, 30(1), 156-178 (2024).
[10] Çelik, B., Set, E., Akdemir, A. O., & Agarwal, P. The Minkowski inequality associated with constant proportional fractional integrals. Khayyam Journal of Mathematics, 10(2), 273-287 (2024).
[11] Bayraktar, B., Butt, S. I., Napoles, J. E., & Rabossi, F. Some new estimates for integral inequalities and their applications. Ukrainian Mathematical Journal, 76(2), 169-191 (2024).