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Spherical Curves with Modified Orthogonal Frame with Torsion

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Abstract. In this paper, we studied the spherical curves according to modified orthogonal frame with torsion in 3 dimensional Euclidean space. We obtained the center, the radius and spherical condition of spherical curves according to the 3 dimensional Euclidean space.

1. Introduction

The theory of curves is one of the most important areas of study in differential geometry.The concept of the curve that Euler defined in plane moved to three-dimensional Euclidean space by Fujiwara (1914)[3]. It is well known from the literature that; in order to examine the geometry of a given curve, Frenet equations belonging to this curve must be known. These equations are also known as the Serret-Frenet equations, and it can be understood whether a curve is planar or a line. Studies on this subject were first made for space curves [1, 2]. Considering that the given curve can also be found on a surface, the geometry of these types of curves has been investigated by many mathematicians on the subject [7, 8]. These investigations have been made especially for curves on a sphere, which are called spherical curves [4, 9, 10]. Wong (1963) stated that a global formulation of the condition for a curve to lie in a sphere [4]. This formülation has taken its place as a necessary and sufficient condition for a curve to lie in a sphere in books written on differential geometry. Wong (1972) reached an explicit characterization of spherical curves [5].Considering the definition of the sphere, it is clear that the sphere is actually related to the given dot product. When the subject is considered from this point of view, it can be thought that spherical curves can have very different characterizations in Euclidean and semi-Euclidean spaces. In this study, the spherical curve studies, which were done according to the Serret-Frenet frame previously defined in Euclidean space, which were done according to modified orthogonal frame previously defined in Euclidean space, were made according to the orthogonal frame modified with torsion, also defined in the Euclidean space [6].

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2. Preliminaries

We initially give the classical basic theorem of space curves in 3 dimensional Euclidean space. We assume that the curve $\beta(u)$ in $\mathbb{C}^3$ is parametrized by arc-length. In addition we suppose that its curvature $\kappa(s)$ never vanish. Then orthonormal frame $\{t, n, b\}$ which satisfies the Frenet-Serret equation is as follows:

$$\begin{pmatrix} t^{'}(s) \\ n^{'}(s) \\ b^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{1}$$

in which t, n, b are the unit tangent, principal normal and binormal vectors, respectively, and $\tau(s)$ is the torsion. Then an orthonormal frame $\{t, n, b\}$ exists satisfying the equation (2.1). Now we assume that the curvature $\kappa(s)$ of β is not identically zero. We define an orthogonal frame $\{T, N, B\}$ by

$$T = \frac{d\beta}{ds}, N = \frac{dT}{ds}, B = T \wedge N$$

in which $\wedge$ denotes the vector product. Then we can give the relation between $\{T, N, B\}$ and $\{t, n, b\}$ as follows:

$$\begin{aligned} T &= t \\ N &= \tau n \\ B &= \tau b \end{aligned} \tag{2}$$

From the definition of $\{T, N, B\}$ or equation (2.2), we can write matrix form as:

$$\begin{pmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa(s)}{\tau(s)} & 0 \\ -\kappa(s)\tau(s) & \frac{\tau^{'}(s)}{\tau(s)} & \tau(s) \\ 0 & -\tau(s) & \frac{\tau^{'}(s)}{\tau(s)} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{3}$$

In addition to, $\{T, N, B\}$ satisfies:

$$\langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = \tau^2$$
$$\langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0$$

in which $\langle , \rangle$ represents the Euclidean inner product [6].

3. Spherical Curves With Modified Orthogonal Frame With Torsion

Definition 3.1. *Let β be in $\mathbb{E}^3$ given by coordinate neighborhood (I, β). If $\beta \subset \mathbb{E}^3$ then β is defined by a spherical curve of $\mathbb{E}^3$ [6].*

Definition 3.2. *The sphere having sufficiently close common four points at $m \in \beta$ the curve $\beta \subset \mathbb{E}^3$ is called the osculating sphere or curvature sphere of the curve β at the point $m \in \beta$ [6].*

Theorem 3.3. *Assume that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). The geometric locus of the centers of the spherical curves with 3-contact points with the curve β providing the modified orthogonal frame with torsion vectors $\{T, N, B\}$ at the point $\beta(s)$, $s \in I$ is*

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

in which

$$m_2 : I \to R, m_2(s) = \frac{1}{\kappa\tau}, \quad \textit{and} \quad m_3(s) = \frac{\pm 1}{\kappa\tau}\sqrt{r^2k^2 - 1}.$$

Proof. Let (I,β) be a coordinate neighborhood, s is arc length parameter. Let also a be the center and, r be the radius of the sphere with 3-contact points with β. From this, let us consider

$$f : I \to R$$

$$s \to f(s) = \langle a-\beta(s), a-\beta(s)\rangle - r^2 \tag{4}$$

Since

$$f(s) = f'(s) = f''(s) = 0 \tag{5}$$

at the point $\beta(s)$, then the sphere

$$S^2 = \left\{x \in \mathbb{E}^3 : \langle x-a, x-a\rangle = r^2\right\}, \text{(x generic point of the sphere)},$$

with the curve β at this point passes sufficiently close three points. Therefore, considering equations (3.1) and (3.2)

$$f(s) = \langle a-\beta(s), a-\beta(s)\rangle - r^2 = 0$$
$$f'(s) = 0 \implies \langle T, a-\beta(s)\rangle = 0$$

is obtained. From this, since $f''(s) = 0$, we get

$$\left\langle T', a-\beta(s)\right\rangle + \langle T, -\beta'(s)\rangle = 0$$

is obtained. Considering equation (2.3) with this, we have

$$\langle N, a-\beta(s)\rangle = \frac{\tau}{\kappa}$$

On the other words, for the base $\{T, N, B\}$,

$$a-\beta(s) = m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s) \tag{6}$$

is obtained. But, from using using equation (3.2), we have

$$m_1(s) = \langle a-\beta(s), T(s)\rangle = 0 \tag{7}$$

and

$$m_2(s)\tau^2 = \langle a-\beta(s), N(s)\rangle \implies m_2 = \frac{1}{\kappa\tau} \tag{8}$$

With the assistance of $f(s) = 0$, we have

$$\langle a-\beta(s), a-\beta(s)\rangle = r^2 \to m_1^2(s) + m_2^2(s)\tau^2 + m_3^2(s)\tau^2 = r^2 \tag{9}$$

Considering equation (3.4) and equation (3.5),we have

$$m_3(s) = \frac{\pm 1}{\kappa\tau}\sqrt{r^2k^2-1} = \lambda \in R \tag{10}$$

Therefore, subtituting equations (3.4), (3.5) and (3.7) into equation (3.3)

$$a(s) = \beta(s) + \frac{1}{\kappa\tau}N(s) \pm \frac{1}{\kappa\tau}\sqrt{r^2k^2-1}B(s)$$

Thus, the proof of the theorem is completed. □

Corollary 3.4. *Assume that β is in $\mathbb{E}^3$ given by coordinate neighborhood (I,β). Then the centers of the spheres with 3-contact points with the β at the points $\beta(s) \in \beta$ lie on a straight line.*

Proof. From Theorem 1, we have

$$a(s) = \beta(s) + \frac{1}{\kappa\tau}N(s) + \lambda B(s)$$

The equation with λ parameter denotes a line which pass through the point $C(s) = \beta(s) + \frac{1}{\kappa\tau}N(s)$ and is parallel to the B. □

Definition 3.5. *The line $a(s) = \beta(s) + \frac{1}{\kappa\tau}N(s) + \lambda B(s)$ is the geometric locus of the centers of the spheres with 3-contact points with the curve at $\beta \subset \mathbb{E}^3$ at the point $m \in \beta$ is called curvature the axis at the point $m \in \beta$ of curve $\beta \subset \mathbb{E}^3$. The point*

$$C(s_0) = \beta(s_0) + \frac{1}{\kappa\tau}N(s_0)$$

on curvature the axis is called curvature the center at the point $m = \beta(s_0)$ of curve $\beta \subset \mathbb{E}^3$.

Theorem 3.6. *Assume that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). If*

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

is the center of the osculating sphere at the point $\beta(s) \in \beta$ then

$$m_2(s) = \frac{1}{\kappa\tau} \quad \text{and} \quad m_3(s) = \frac{-\kappa'}{\kappa^2\tau^2}$$

Proof. The proof of the theorem is similar to the proof of Theorem 1. The osculating sphere with the curve β have sufficiently close common four points. So, from $f''(s) = 0$ in equation (3.2) thus $f'''(s) = 0$. Then we get

$$\frac{\kappa'}{\kappa} - \frac{\tau'}{\tau} + \frac{\kappa}{\tau}(-\kappa\tau\langle T, a - \beta(s)\rangle + \frac{\tau'}{\tau}\langle N, a - \beta(s)\rangle + \tau\langle B, a - \beta(s)\rangle) = 0$$

Considering equations (3.4) and (3.5) in the last equality, we get

$$\langle B, a - \beta(s)\rangle = \frac{-\kappa'}{\kappa^2}$$

or

$$m_3(s) = \frac{-\kappa'}{\kappa^2\tau^2}$$

□

Corollary 3.7. *Suppose that β is in $\mathbb{E}^3$ given with coordinate neighborhood (I, β). The radius of the osculating sphere is:*

$$r = \sqrt{(m_2^2(s) + m_3^2(s))\tau^2} = \sqrt{\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2}$$

Proof. From Theorem 1,

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

Therefore, we have

$$r = \left\|a - \beta(s)\right\| = \sqrt{(m_2^2\langle N, N\rangle + m_3^2\langle B, B\rangle} = \sqrt{(m_2^2(s) + m_3^2(s))\tau^2} = \sqrt{\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2}$$

□

Theorem 3.8. *Let S_0^2 be a sphere centered at zero and also $\beta \subset S_0^2$ be a spherical curve. In this case,*

$$-m_1(s) = \langle \beta(s), T \rangle, \quad -m_2(s) = \frac{\langle \beta(s), N \rangle}{\tau^2} \quad \textit{and} \quad -m_3(s) = \frac{\langle \beta(s), B \rangle}{\tau^2}$$

Proof. Since $\beta \subset S_0^2$ for all $s \in I$, and r is radius, then we have

$$\overrightarrow{0} = \beta(s) + m_1 T + m_2 N + m_3 B$$

and

$$\langle \beta(s), \beta(s) \rangle = r^2$$

Thus, by differentiation of the above equations with respect to s we have

$$-m_1 = \langle \beta(s), T \rangle = 0$$

by differentiation of the above equations with respect to s we have

$$\langle \beta(s), N \rangle = \frac{-\tau}{\kappa}$$

and

$$-m_2(s) = \frac{\langle \beta(s), N \rangle}{\tau^2}$$

and

$$\langle \beta(s), B \rangle = \frac{\kappa'}{\kappa^2}$$

Thus, since $-m_3(s) = \frac{\kappa'}{\kappa^2 \tau^2}$, we can write the last equality as

$$-m_3(s) = \frac{\langle \beta(s), B \rangle}{\tau^2}$$

□

Theorem 3.9. $S_0^2 \subset \mathbb{E}^3$ *be a sphere whose center is at the origin. If β is a curve on S_0^2, then the osculating sphere of the curve β at every point is S_0^2.*

Proof. suppose thet the curve β with (I, β) neighbouring coordinate such that $s \in I$ is arclength parameter. By Theorem 2

$$a(s) = \beta(s) + m_2(s)N(s) + m_3(s)B(s)$$

By Theorem 3, this expression can be written as

$$a(s) = \beta(s) - \frac{\langle \beta(s), N \rangle}{\tau^2} N(s) - \frac{\langle \beta(s), B \rangle}{\tau^2} B(s)$$

Since $\langle \beta(s), T \rangle = 0$, we get

$$a(s) = \beta(s) - \frac{\langle \beta(s), N \rangle}{\tau^2} N(s) - \frac{\langle \beta(s), B \rangle}{\tau^2} B(s)$$

Thus we get

$$a = \beta(s) - \beta(s) = 0$$

This completes the proof of the theorem. □

Theorem 3.10. *Let $\beta : I \to E^3$ be a given curve whose $\tau \neq 0$ for all $s \in I$ and let $m_3(s) \neq 0$. The radius of the osculating sphere at the point $\beta(s)$ is constant for all $s \in I$ if and only if the centers of the osculating sphere are the same point.*

Proof. $\Longrightarrow$: By Corollary 2, we can write as follows

$$r^2 = (m_2^2(s) + m_3^2(s))\tau^2$$

Since r=constant, by differentiation of with this equation respect to s, we have

$$(2m_2m_2' + 2m_3m_3')\tau^2 + 2\tau\tau'(m_2^2 + m_3^2) = 0$$

or

$$m_3' + \frac{\tau'}{\tau}m_3 = \frac{-\tau'}{\tau}\frac{m_2}{m_3}m_2 - \frac{m_2}{m_3}m_2'$$

Inverting values $m_2 = \frac{1}{\kappa\tau}$,$m_3 = \frac{-\kappa'}{\kappa^2\tau^2}$ and $m_2' = \frac{-\kappa'\tau+\tau'\kappa}{\kappa^2\tau^2}$ in right side of the last equality, we obtain

$$m_3' + \frac{\tau'}{\tau}m_3 = \frac{-1}{\kappa} = -\tau m_2$$

Finally, since $m_2 = \frac{1}{\kappa\tau}$, we get

$$m_3' + \frac{\tau'}{\tau}m_3 + \tau m_2 = 0 \tag{11}$$

Otherwise for base $\{T, N, B\}$ we get

$$a(s) = \beta(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s)$$

From derivative with respect to s of the last equality, we get

$$a'(s) = (1 - \kappa\tau m_2)T + (m_2' + m_2\frac{\tau'}{\tau} - \tau m_3)N + (\tau m_2 + m_3' + \frac{\tau'}{\tau}m_3)B \tag{12}$$

Inverting values $m_2 = \frac{1}{\kappa\tau}$,$m_3 = \frac{-\kappa'}{\kappa^2\tau^2}$ and $m_2' = \frac{-\kappa'\tau+\tau'\kappa}{\kappa^2\tau^2}$ in right side of the last equality, we obtain

$$a'(s) = (\tau m_2 + m_3' + \frac{\tau'}{\tau}m_3)B$$

So by Equation (3.8) $m_3' + \frac{\tau'}{\tau}m_3 + \tau m_2 = 0$ we find $a'(s) = 0$ and so $a = constant$ for all $s \in I$.
Conversely, suppose that $a(s) = constant$ for all $s \in I$. According to the equation

$$\langle a(s) - \beta(s), a(s) - \beta(s)\rangle = r^2,$$

taking differentiation of this equation with respect to s,we made

$$r(s)r'(s) = 0$$

Here, either $r(s) = 0$ or $r'(s) = 0$. If $r(s) = 0$, then by Corollary 2, we have

$$(m_2^2(s) + m_3^2(s))\tau^2 = 0, \tau \neq 0$$

or

$$m_2^2(s) = -m_3^2(s) = 0$$

But this contradicts the theorem. So $r'(s) = 0$. Thus, $r(s)$ is constant for all $s \in I$. □

Theorem 3.11. *Let $\beta : I \to E^3$ be a curve such that $m_3(s) \neq 0$, for all $s \in I$ and $\tau \neq 0$. Then, the curve β lies on a sphere if and only if the centers of the osculating spheres of the curve β are all the same point.*

Proof. Let β be a curve on S_b^2 which have the radius r and centered at any point b. By Theorem 4, the proof is clear. Conversely, let the centers of the osculating curve be the point b in $\beta(s) \in \beta$ all $s \in I$. Then by Theorem 5 all osculating spheres have the same radius r. Therefore

$$d(\beta(s), b) = r$$

for all $s \in I$. This completes the proof of the theorem. □

Theorem 3.12. *Let the curve β in $\mathbb{E}^3$ be given with coordinate neighborhood (I, β) and $m_3(s) \neq 0$, $\tau \neq 0$ such that s is a arclength parameter, then, β is a spherical curve if and only if*

$$(\frac{-\kappa^{'}}{\kappa^2\tau^2})^{'} - \frac{\tau^{'}\kappa^{'}}{\kappa^2\tau^3} + \frac{1}{\kappa} = 0$$

Proof. Let β be a spherical curve. By Theorem 6 , for all $s \in I$, the center $a(s)$ of the osculating spheres are constant. Moreover, the equation (3.8) yields

$$m_3^{'} + \frac{\tau^{'}}{\tau} m_3 + \tau m_2 = 0$$

or

$$(\frac{-\kappa^{'}}{\kappa^2\tau^2})^{'} - \frac{\tau^{'}\kappa^{'}}{\kappa^2\tau^3} + \frac{1}{\kappa} = 0$$

Conversely, let $m_3^{'} + \frac{\tau^{'}}{\tau} m_3 + \tau m_2 = 0$ By Theorem 5 and $a^{'}(s) = 0$. Therefore $a(s) = constant$. So by Theorem 6, β is a spherical curve. □

Example 3.13. *Let the curve β such that $c = 2\sqrt{ab}$ and $r = a + b$.*

$$\beta(t) = (acost + bcos3t, asint - bsin3t, csin2t)$$

We find radius and center of the osculating sphere at the point $t = 0$. Since

$$\beta'(t) = (-asint - 3bsin3t, acost - 3bcos3t, 2ccos2t)$$

$$\left\|\beta'(t)\right\| = \sqrt{(a + 3b)^2 + (ccos2t)^2}.$$

For $t = 0$, $a = 1$, $b = 1$, $c = 2$, $r = 2$, Since $\left\|\beta'(0)\right\| = 2\sqrt{5}$ t is a arbitrary parameter. From this, we can find $\{t, n, b\}$ Frenet vectors.

$$t = \frac{\beta^{'}}{\left\|\beta'\right\|} = (0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

$$b = \frac{\beta^{'}(0) \wedge \beta^{''}(0)}{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|} = (0, \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$$

$$n = b \wedge t = (-1, 0, 0)$$

The curvature and torsion of the curve β are as follows:

$$\kappa = \frac{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|}{\left\|\beta^{'}(0)\right\|^3} = \frac{1}{2}$$

$$\tau = \frac{det(\beta', \beta'', \beta''')}{\left\|\beta^{'}(0) \wedge \beta^{''}(0)\right\|^2} = \frac{-9}{25}$$

From this, we can find $\{T, N, B\}$ Frenet vectors of the modified orthogonal frame with torsion.

$$T = t = (0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$$

$$N = \tau n = (\frac{9}{25}, 0, 0)$$

$$B = \tau b = (0, \frac{18}{25\sqrt{5}}, \frac{9}{25\sqrt{5}})$$

We can find coordinates of the center of the osculating sphere at the point $\beta(0)$. From $a(t) = \beta(t) + m_2(t)N(t)$, $a(0) = (4, 0, 0)$. We can find the radius of the osculating sphere at the point $\beta(0)$ as seen in Figure 1 .

$$r = \sqrt{(m_2^2(t) + m_3^2(t))\tau^2} = 2$$

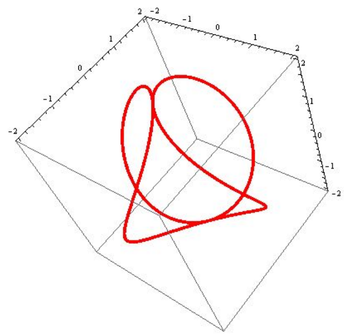

Figure.1. β spherical curve .

4. Conclusion

In this study , we obtained the center, the radius and spherical condition of spherical curves according to modified orthogonal frame with torsion. An example of a spherical curve is given according to this frame. The results obtained can also be found according to other frame, for future works.

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