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Fractional Newton-Type Inequalities Involving s-Convex Functions

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Abstract. In this paper, we present a novel method to examine Newton-type inequalities for s-convex functions using Riemann-Liouville fractional integrals. Specifically, we establish new fractional Newton-type inequalities by employing s-convex functions. The s-convexity property is extensively utilized to derive these inequalities. Additionally, we make use of the Holder inequality and the power-mean inequality to further strengthen our results. These new inequalities offer a more generalized perspective compared to classical inequalities, highlighting the significant role of s-convexity in modern mathematical analysis.

1. Introduction

The fundamental concept behind Simpson's second rule is derived from the three-point Newton-Cotes quadrature rule. Typically, results involving three-step quadratic kernel computations are referred to as Newton-type results. These findings are recognized as Newton-type inequalities in the mathematical literature. Various mathematicians have studied Newton-type inequalities. For example, several Newton-type inequalities for functions whose second derivatives are convex are established in [1]. Additionally, Newton-type inequalities have been developed for harmonic convex and p-harmonic convex functions in [2] and [3], respectively. In [4], some Newton-type inequalities are obtained using post-quantum integrals. Furthermore, [5] presents several Newton-type inequalities for quantum differentiable convex functions. For more related research, see [6–9].

The popularity of fractional calculus has increased significantly in recent years due to its wide range of applications across various scientific fields. Given the importance of fractional calculus, several operators for fractional integrals have been studied. For instance, in [10], some Newton-type inequalities are derived for functions whose first derivatives, in absolute value raised to a certain power, are arithmetically-harmonically convex. Additionally, Newton-type inequalities are established using Riemann-Liouville fractional integrals for differentiable convex functions, and some Riemann-Liouville fractional Newton-type inequalities are provided for functions of bounded variation in [11]. Furthermore, several Newton-type inequalities are formulated using the well-known Riemann-Liouville fractional integrals for differentiable convex functions in [12]. For more details on related topics, please refer to the articles [14–20]. The paper is organized into five sections, each providing a comprehensive exploration of different aspects of fractional calculus and Newton-type inequalities.

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In this section, we provide an overview of the motivations and significance of the study, along with a brief literature review on fractional calculus and Newton-type inequalities. Section 2 presents the fundamental definitions and preliminaries necessary for understanding fractional calculus. This section also reviews relevant research that has been conducted in this field, laying the groundwork for the subsequent discussions. Section 3 delves into the core contributions of the paper, where we derive several fractional Newton-type inequalities for differentiable s-convex functions using Riemann-Liouville fractional integrals. These results not only extend existing inequalities but also introduce new perspectives in the analysis of s-convex functions. Section 4concludes the paper by presenting the implications of the derived Newton-type inequalities. We explore how these findings might influence future research directions in fractional calculus and related areas. This section also reflects on the potential applications of Newton-type inequalities in various scientific and mathematical contexts.

2. Preliminaries

A Simpson-type inequality is a variation of an inequality derived from Simpson's rule. It can be formulated as follows:

i. Simpson's 1/3 Rule, also known as Simpson's Quadrature Formula:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. Simpson's 3/8 Rule, also referred to as Simpson's Second Formula or the Newton-Cotes Quadrature Formula:

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

The three-point Simpson-type inequality is one of the most widely used Newton-Cotes quadrature formulas and is stated as follows:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a function that is four times differentiable and continuous on (a,b), and let $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x),dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

The Simpson's 3/8 rule is another classical closed-type quadrature formula and can be expressed as follows:

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a function that is four times differentiable and continuous on (a,b), and let $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x),dx\right| \le \frac{1}{6480}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Definition 2.3 (See [21]). *Let I be an interval of real numbers. A function $f:I\to\mathbb{R}$ is said to be convex if the following inequality holds for all $x,y\in I$ and for all $t\in[0,1]$:*

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y).$$

Conversely, the function f is said to be concave on I if the above inequality is reversed for all $t\in[0,1]$ and $x,y\in I$.

Definition 2.4 (See [22]). *Let $f : [0,\infty) \to \mathbb{R}$ be a function and let $0 < s \leq 1$. The function f is said to be s-convex in the second sense if for all $\mu, \nu \geq 0$ with $\mu + \nu = 1$, the following inequality holds:*

$$f(\mu x + \nu y) \leq \mu^s f(x) + \nu^s f(y) \tag{1}$$

for all $x, y \in [0,\infty)$.

Remark 2.5. *If we take $s = 1$ in Definition 2.4, then it reduces to the classical definition of convexity given in Definition 2.3.*

Definition 2.6 (See [23, 24]). *Let $f \in L_1[a,b]$, where $a, b \in \mathbb{R}$ and $a < b$. The Riemann-Liouville fractional integrals of order $\alpha > 0$, denoted by $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$, are defined as follows:*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \text{ for } x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \text{ for } x < b,$$

Here, Γ denotes the Gamma function, which is defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

The following integral identity, previously obtained by Hezenci and Budak [13], is fundamental for presenting the main results of this paper:

Lemma 2.7. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on (a,b) such that $f' \in L_1[a,b]$. Then, the following identity holds:*

$$\begin{aligned} &\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b)\right] \\ &= \frac{b-a}{4}\left[I_1 + I_2\right], \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{2}{3}} \left(t^{\alpha} - \frac{1}{4}\right)\left[f'\left(\frac{t}{2}b + \frac{2-t}{2}a\right) - f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right] dt, \\ I_2 &= \int_{\frac{2}{3}}^1 \left(t^{\alpha} - 1\right)\left[f'\left(\frac{t}{2}b + \frac{2-t}{2}a\right) - f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right] dt. \end{aligned}$$

Using this identity and the properties of s-convex functions, along with Hölder inequality and the power-mean inequality, we will derive new inequalities in the subsequent section.

3. Fractional Newton-Type Inequalities Involving s-Convex Functions

In this section, some new fractional Newton-type inequalities will be obtained using s-convex functions.

Theorem 3.1. *Assume that the conditions of Lemma 2.7 are satisfied and that the function $|f'|$ is s-convex on the interval $[a,b]$. Then, the following fractional Newton-type inequality can be established:*

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b)\right]\right| \tag{2}$$

$$\leq \frac{b-a}{4}\left(\Upsilon_1(\alpha,s)+\Upsilon_2(\alpha,s)+\Upsilon_3(\alpha,s)+\Upsilon_4(\alpha,s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

Here,

$$\begin{aligned}
\Upsilon_1(\alpha,s) &= \int_0^{\frac{2}{3}}\left(\frac{t}{2}\right)^s\left|t^\alpha-\frac{1}{4}\right|dt \\
&= \begin{cases} \frac{1}{2^s}\left[\left(\frac{1}{4}\right)^{\frac{s+1}{\alpha}}\frac{1}{2(s+1)}-\left(\frac{1}{4}\right)^{\frac{\alpha+s+1}{\alpha}}\frac{2}{\alpha+s+1}+\left(\frac{2}{3}\right)^{\alpha+s+1}\frac{1}{\alpha+s+1}-\left(\frac{2}{3}\right)^{s+1}\frac{1}{4(s+1)}\right], & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}, \\ \frac{1}{2^s}\left[\left(\frac{2}{3}\right)^{s+1}\left(\frac{1}{4(s+1)}-\left(\frac{2}{3}\right)^{\alpha+s+1}\frac{1}{\alpha+s+1}\right)\right], & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha \end{cases}
\end{aligned}$$

$$\Upsilon_2(\alpha,s)=\int_0^{\frac{2}{3}}\left(\frac{2-t}{2}\right)^s\left|t^\alpha-\frac{1}{4}\right|dt,$$

$$\Upsilon_3(\alpha,s)=\int_{\frac{2}{3}}^{1}\left(\frac{t}{2}\right)^s(1-t^\alpha)\,dt=\frac{1}{2^s}\left[\frac{1}{s+1}\left(1-\left(\frac{2}{3}\right)^{s+1}\right)+\frac{1}{\alpha+s+1}\left(1-\left(\frac{2}{3}\right)^{\alpha+s+1}\right)\right],$$

and

$$\Upsilon_4(\alpha,s)=\int_{\frac{2}{3}}^{1}\left(\frac{2-t}{2}\right)^s(1-t^\alpha)\,dt.$$

Proof. Consider the absolute value in Lemma 2.7. Then we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right| \qquad (3)\\
&\leq \frac{b-a}{4}\left\{\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.\\
&\quad\left.+\int_{\frac{2}{3}}^{1}\left|t^\alpha-1\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)-f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right\}.
\end{aligned}$$

Utilizing the s-convexity of $\left|f'\right|$, the following can be derived:

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|\\
&\leq \frac{b-a}{4}\left\{\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|\left[\left(\frac{t}{2}\right)^s\left|f'(b)\right|+\left(\frac{2-t}{2}\right)^s\left|f'(a)\right|+\left(\frac{t}{2}\right)^s\left|f'(a)\right|+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|\right]dt\right.\\
&\quad\left.+\int_{\frac{2}{3}}^{1}(1-t^\alpha)\left[\left(\frac{t}{2}\right)^s\left|f'(b)\right|+\left(\frac{2-t}{2}\right)^s\left|f'(a)\right|+\left(\frac{t}{2}\right)^s\left|f'(a)\right|+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|\right]dt\right\}
\end{aligned}$$

$$= \frac{b-a}{4}\left(\Upsilon_1(\alpha,s)+\Upsilon_2(\alpha,s)+\Upsilon_3(\alpha,s)+\Upsilon_4(\alpha,s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

□

Remark 3.2. *If we choose $s = 1$ in Theorem 3.1, then we obtain following inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \le \frac{b-a}{4}\left(\Phi_1(\alpha)+\Phi_2(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{4}$$

where

$$\Phi_1(\alpha) = \int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|dt = \begin{cases} \frac{2\alpha}{\alpha+1}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{1}{6}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}, \\ \frac{1}{6}-\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha, \end{cases}$$

$$\Phi_2(\alpha) = \int_{\frac{2}{3}}^{1}\left(1-t^{\alpha}\right)dt = \frac{1}{3}-\frac{1}{\alpha+1}+\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}.$$

This inequality was given by Budak and Hezenci in [13].

Corollary 3.3. *If $\alpha = 1$ is chosen in Theorem 3.1,, then we obtain following inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{4}\left(\Upsilon_1(1,s)+\Upsilon_2(1,s)+\Upsilon_3(1,s)+\Upsilon_4(1,s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{5}$$

Here,

$$\Upsilon_1(1,s) = \int_0^{\frac{2}{3}}\left(\frac{t}{2}\right)^s\left|t-\frac{1}{4}\right|dt = \frac{1}{2^s}\left[\frac{1}{2^{2s+3}(s+1)(s+2)}\right]+\left(\frac{2}{3}\right)^{s+1}\left(\frac{5s+2}{12(s+1)(s+2)}\right)$$

$$\begin{aligned}\Upsilon_2(1,s) &= \int_0^{\frac{2}{3}}\left(\frac{2-t}{2}\right)^s\left|t-\frac{1}{4}\right|dt \\ &= \frac{1}{2^s}\left[\frac{4^{s+2}}{3^{s+2}}-\frac{2\cdot 7^{s+1}}{4^{s+2}}+\frac{4^s}{3^{s-1}(s+1)}+\frac{s\left(2^{s+3}-2^{s-1}\right)+2^{s+2}+2^{s+3}-2^s}{(s+1)(s+2)}\right],\end{aligned}$$

$$\Upsilon_3(1,s) = \int_{\frac{2}{3}}^{1}\left(\frac{t}{2}\right)^s(1-t)\,dt = \frac{1}{2^s}\left[\frac{1}{s+1}\left(1-\left(\frac{2}{3}\right)^{s+1}\right)+\frac{1}{s+2}\left(1-\left(\frac{2}{3}\right)^{s+2}\right)\right],$$

and

$$\Upsilon_4(1,s) = \int_{\frac{2}{3}}^{1}\left(\frac{2-t}{2}\right)^s(1-t)\,dt = \frac{1}{2^s}\left[\frac{1}{(s+1)(s+2)}+\frac{4^{s+2}}{3^{s+2}(s+2)}-\frac{4^{s+1}}{3^{s+1}(s+1)}\right].$$

Remark 3.4. *By choosing $\alpha = 1$, $s = 1$ in Theorem 3.1, we obtain the following inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{25(b-a)}{576}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which was derived by Sitthiwirattham et al. in [11, Remark 3].

Theorem 3.5. *Suppose the conditions of Lemma 2.7 are satisfied and the function $\left|f'\right|^q$, where $q > 1$, is s-convex on the interval $[a,b]$. Then, the following Newton-type inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right| \tag{6}$$

$$\le \frac{b-a}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\frac{1}{2^s(s+1)}\left(\frac{2}{3}\right)^{s+1}\left|f'(b)\right|^q+\frac{1}{2^s(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^s(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]\left|f'(b)\right|^q+\frac{1}{2^s(s+1)}\left(\frac{2}{3}\right)^{s+1}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{2}{3}}^1(1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left[\frac{1}{2^s(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]\left|f'(b)\right|^q\right.$$

$$\left.+\frac{1}{2^s(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right]+\left[\frac{1}{2^s(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]\left|f'(a)\right|^q\right.$$

$$\left.\left.+\frac{1}{2^s(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Utilizing Hölder's inequality in (3), we obtain

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right|$$

$$\le \frac{b-a}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^1\left|t^\alpha-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^1\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}|t^{\alpha}-1|^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg\}.$$

Given the properties of s-convexity for $|f'|^q$, the following can be easily derived

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$

$$\leq\frac{b-a}{4}\left\{\left(\int_{0}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{2}{3}}\left(\left(\frac{t}{2}\right)^{s}|f'(b)|^{q}+\left(\frac{2-t}{2}\right)^{s}|f'(a)|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{0}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{2}{3}}\left(\left(\frac{t}{2}\right)^{s}|f'(a)|^{q}+\left(\frac{2-t}{2}\right)^{s}|f'(b)|^{q}\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left(\left(\frac{t}{2}\right)^{s}|f'(b)|^{q}+\left(\frac{2-t}{2}\right)^{s}|f'(a)|^{q}\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left(\left(\frac{t}{2}\right)^{s}|f'(a)|^{q}+\left(\frac{2-t}{2}\right)^{s}|f'(b)|^{q}\right)dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{b-a}{4}\left\{\left(\int_{0}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\frac{1}{2^{s}(s+1)}\left(\frac{2}{3}\right)^{s+1}|f'(b)|^{q}+\frac{1}{2^{s}(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]|f'(a)|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2^{s}(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]|f'(b)|^{q}+\frac{1}{2^{s}(s+1)}\left(\frac{2}{3}\right)^{s+1}|f'(a)|^{q}\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{2}{3}}^{1}(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\frac{1}{2^{s}(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]|f'(b)|^{q}\right.$$

$$\left.+\frac{1}{2^{s}(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]|f'(a)|^{q}\right)^{\frac{1}{q}}\Bigg]$$

$$+\left[\frac{1}{2^{s}(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]|f'(a)|^{q}\right.$$

$$\left.\left.+\frac{1}{2^{s}(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]|f'(b)|^{q}\right)^{\frac{1}{q}}\right]\right\}$$

□

Remark 3.6. *By setting $s = 1$ in Theorem 3.5, we derive inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \tag{7}$$
$$\leq \frac{b-a}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+5|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left[\left(\frac{5|f'(b)|^q+7|f'(a)|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{5|f'(a)|^q+7|f'(b)|^q}{36}\right)^{\frac{1}{q}}\right]\right\}$$

as presented by Hezenci and Budak in [13].

Corollary 3.7. *If we choose $\alpha = 1$ in Theorem 3.5, then we obtain following inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{8}$$
$$\leq \frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\frac{1}{2^s(s+1)}\left(\frac{2}{3}\right)^{s+1}|f'(b)|^q+\frac{1}{2^s(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]|f'(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2^s(s+1)}\left[2^{s+1}-\left(\frac{4}{3}\right)^{s+1}\right]|f'(b)|^q+\frac{1}{2^s(s+1)}\left(\frac{2}{3}\right)^{s+1}|f'(a)|^q\right)^{\frac{1}{q}}\right]$$
$$+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{2^s(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]|f'(b)|^q\right.$$
$$\left.+\frac{1}{2^s(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]|f'(a)|^q\right)^{\frac{1}{q}}\right]+\left[\frac{1}{2^s(s+1)}\left[1-\left(\frac{2}{3}\right)^{s+1}\right]|f'(a)|^q\right.$$
$$\left.\left.+\frac{1}{2^s(s+1)}\left[\left(\frac{4}{3}\right)^{s+1}-1\right]|f'(b)|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Remark 3.8. *If $\alpha = 1$ and $s = 1$ are chosen in Theorem 3.5, the resulting inequality is as follows*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{b-a}{4}\left\{\left(\frac{1}{p+1}\left[\left(\frac{1}{4}\right)^{p+1}+\left(\frac{5}{12}\right)^{p+1}\right]\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\frac{|f'(b)|^q+5|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+5|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right]$$

$$+\left(\frac{1}{p+1}\left(\frac{1}{3}\right)^{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{5\left|f'(b)\right|^{q}+7\left|f'(a)\right|^{q}}{36}\right)^{\frac{1}{q}}+\left(\frac{5\left|f'(a)\right|^{q}+7\left|f'(b)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which was proved by Hezenci and Budak in [13].

Theorem 3.9. *If the conditions of Lemma 2.7 are satisfied and the function* $\left|f'\right|^{q}$, *where* $q \geq 1$, *is s-convex on* $[a,b]$, *then the following Newton-type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right| \qquad (9)\\
&\leq \frac{b-a}{4}\Bigg\{(\Phi_1(\alpha))^{1-\frac{1}{q}}\Bigg[\left[\Upsilon_1(\alpha,s)\left|f'(b)\right|^{q}+\Upsilon_2(\alpha,s)\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\\
&+\left[\Upsilon_2(\alpha,s)\left|f'(a)\right|^{q}+\Upsilon_1(\alpha,s)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\Bigg]\\
&+(\Phi_2(\alpha))^{1-\frac{1}{q}}\Bigg[\left[\Upsilon_3(\alpha,s)\left|f'(b)\right|^{q}+\Upsilon_4(\alpha,s)\left|f'(a)\right|^{q}\right]^{\frac{1}{q}}\\
&+\left[\Upsilon_4(\alpha,s)\left|f'(a)\right|^{q}+\Upsilon_3(\alpha,s)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}\Bigg]\Bigg\},
\end{aligned}$$

where $\Upsilon_1(\alpha,s)$, $\Upsilon_2(\alpha,s)$, $\Upsilon_3(\alpha,s)$, $\Upsilon_4(\alpha,s)$ *are given in Theorem 3.1, and* $\Phi_1(\alpha)$, $\Phi_2(\alpha)$ *are defined in Remark 3.2.*

Proof. Utilizing (3) in conjunction with the power-mean inequality, we derive

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\
&\leq \frac{b-a}{4}\Bigg\{\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-1\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-1\right|\left|f'\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-1\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-1\right|\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Given that $\left|f'\right|^{q}$ is s-convex, we can obtain

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|\\
&\leq \frac{b-a}{4}\Bigg\{\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{4}\right|\left[\left(\frac{t}{2}\right)^{s}\left|f'(b)\right|^{q}+\left(\frac{2-t}{2}\right)^{s}\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|t^\alpha-\frac{1}{4}\right|\left[\left(\frac{t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^1(1-t^\alpha)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^1\left|t^\alpha-1\right|\left[\left(\frac{t}{2}\right)^s\left|f'(b)\right|^q+\left(\frac{2-t}{2}\right)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^1(1-t^\alpha)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^1\left|t^\alpha-1\right|\left[\left(\frac{t}{2}\right)^s\left|f'(a)\right|^q+\left(\frac{2-t}{2}\right)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}.$$

□

Remark 3.10. *In Theorem 3.9, by choosing $s=1$, we derive inequality*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}-}f(a)+J^\alpha_{\frac{a+b}{2}+}f(b)\right]\right| \tag{10}$$

$$\leq \frac{b-a}{4}\Bigg\{(\Phi_1(\alpha))^{1-\frac{1}{q}}\Bigg[\left[\Upsilon_1(\alpha,1)\left|f'(b)\right|^q+\Upsilon_2(\alpha,1)\left|f'(a)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\Upsilon_2(\alpha,1)\left|f'(a)\right|^q+\Upsilon_1(\alpha,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\Bigg]$$

$$+(\Phi_2(\alpha))^{1-\frac{1}{q}}\Bigg[\left[\Upsilon_3(\alpha,1)\left|f'(b)\right|^q+\Upsilon_4(\alpha,1)\left|f'(a)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\Upsilon_4(\alpha,1)\left|f'(a)\right|^q+\Upsilon_3(\alpha,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\Bigg]\Bigg\},$$

where $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$ are given in Remark 3.2 and

$$\Upsilon_1(\alpha,1)=\int_0^{\frac{2}{3}}\frac{t}{2}\left|t^\alpha-\frac{1}{4}\right|dt=\begin{cases}\frac{\alpha}{2(\alpha+2)}\left(\frac{1}{4}\right)^{1+\frac{2}{\alpha}}+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}-\frac{1}{36}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\[2ex] \frac{1}{36}-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\end{cases}$$

$$\Upsilon_2(\alpha,1)=\int_0^{\frac{2}{3}}\frac{2-t}{2}\left|t^\alpha-\frac{1}{4}\right|dt=\begin{cases}\frac{2\alpha}{\alpha+1}\left(\frac{1}{4}\right)^{1+\frac{1}{\alpha}}+\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1} \\ -\frac{\alpha+1}{2(\alpha+2)}\left(\frac{1}{4}\right)^{1+\frac{2}{\alpha}}-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}-\frac{5}{36}, & 0<\alpha<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)},\\[2ex] \frac{5}{36}-\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}, & \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha,\end{cases}$$

$$\Upsilon_3(\alpha,1)=\int_{\frac{2}{3}}^1\frac{t}{2}(1-t^\alpha)\,dt=\frac{5}{36}-\frac{1}{2(\alpha+2)}+\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2},$$

$$\Upsilon_4(\alpha,1)=\int_{\frac{2}{3}}^1\frac{2-t}{2}(1-t^\alpha)\,dt=\frac{7}{36}-\frac{\alpha+3}{2(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+1}\left(\frac{2}{3}\right)^{\alpha+1}-\frac{1}{2(\alpha+2)}\left(\frac{2}{3}\right)^{\alpha+2}.$$

This inequality is given by Hezenci and Budak in [13].

Remark 3.11. *When $\alpha = 1$ in Theorem 3.9, we obtain the following inequality*

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{11}$$
$$\leq \frac{b-a}{4}\left\{\left(\frac{17}{144}\right)^{1-\frac{1}{q}}\left[\left[\Upsilon_1(1,s)\left|f'(b)\right|^q + \Upsilon_2(1,s)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.\right.$$
$$\left. + \left[\Upsilon_2(1,s)\left|f'(a)\right|^q + \Upsilon_1(1,s)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]$$
$$+ \left(\frac{1}{18}\right)^{1-\frac{1}{q}}\left[\left[\Upsilon_3(1,s)\left|f'(b)\right|^q + \Upsilon_4(1,s)\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.\left. + \left[\Upsilon_4(1,s)\left|f'(a)\right|^q + \Upsilon_3(1,s)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right]\right\},$$

where $\Upsilon_1(1,s)$, $\Upsilon_2(1,s)$, $\Upsilon_3(1,s)$, and $\Upsilon_4(1,s)$ are given in Corollary 3.3.

Remark 3.12. *By setting $\alpha = 1$ and $s = 1$ in Theorem 3.9, we establish the following inequality*

$$\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{b-a}{72}\left\{\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{251\left|f'(b)\right|^q + 973\left|f'(a)\right|^q}{576}\right)^{\frac{1}{q}} + \left(\frac{251\left|f'(a)\right|^q + 973\left|f'(b)\right|^q}{576}\right)^{\frac{1}{q}}\right]\right.$$
$$\left. + \left[\left(\frac{7\left|f'(b)\right|^q + 11\left|f'(a)\right|^q}{18}\right)^{\frac{1}{q}} + \left(\frac{7\left|f'(a)\right|^q + 11\left|f'(b)\right|^q}{18}\right)^{\frac{1}{q}}\right]\right\},$$

which was given by Hezenci and Budak in [13].

4. Summary & concluding remarks

In this manuscript, several Newton-type inequalities are established for s-convex functions using Riemann-Liouville fractional integrals. Firstly, an integral identity is presented, which is essential for proving the main findings of the paper. Secondly, various Newton-type inequalities are derived for differentiable s-convex functions by employing Riemann-Liouville fractional integrals.

The ideas and methods introduced in our results, relating Riemann-Liouville fractional integrals to Newton-type inequalities, could pave the way for further research in this field. Future studies might explore developments or extensions of our findings by utilizing alternative fractional integral operators or investigating different classes of convex functions. Additionally, the framework of quantum calculus can be applied to obtain various Newton-type inequalities for different convex function classes.

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