TURKISH JOURNAL OF SCIENCE
VOLUME 9, ISSUE 1, 32–44
ISSN: 2587–0971

https://www.tjoscience.com

A Note on Semi-Slant Lightlike Submanifolds of PNsR-Manifolds

Tuba ACET[a]

[a]*Faculty of Arts and Sciences, Department of Mathematics, İnönü University, Malatya, Türkiye*

Abstract. The aim of this paper is to study semi-slant submanifolds of poly-Norden semi-Riemannian manifolds (*PNsR*-manifolds). Also, we obtain some results with non-trivial examples of such submanifolds.

1. Introduction

It is well known that lightlike submanifolds differs noticable from their non-degenerate counterparts, because of degeneracy of the induced metric. Such differences results from the fact that tangent and normal bundle have a non-trivial intersection. This theory is developed by K. L. Duggal and A. Bejancu [1] (see also [2]). Then the study of lightlike submanifolds have been extensively investigated ([3–5]).

In [6], as a generalization of totally real submanifolds and complex submanifolds slant submanifolds of almost Hermitian manifolds introduced by B.Y. Chen. Then this theory was extended different manifold. Semi-slant submanifolds in almost Hermitian manifolds were introduced by N. Papagiuc [7]. Semi-slant submanifolds in Sasakian manifolds were studied by J. L. Cabrerizo [8] (see also [9–11]).

By use of generalization of golden mean, V.W. Spinadel introduced metallic structure [12]. Let ρ_1 and ρ_2 be positive integers. Thus, members of the metallic means family are positive solution

$$x^2 - \rho_1 x - \rho_2 = 0,$$

and this number, which are known $(\rho_1, \rho_2)-$metallic numbers denoted by [13]

$$\sigma_{\rho_1,\rho_2} = \frac{\rho_1 + \sqrt{\rho_1^2 + 4\rho_2}}{2}.$$

A metallic manifold has a tensor field $\tilde{J}$ such that the equality $\tilde{J}^2 = \rho_1\tilde{J} + \rho_2 I$ is satisfied, where the eigenvalues of automorphism $\tilde{J}$ of the tangent bundle are σ_{ρ_1,ρ_2} and $\rho_1 - \sigma_{\rho_1,\rho_2}$ [13]. Metallic structure on the ambient manifold provides useful results on the submanifolds, since it is an impotant tool while examining of submanifolds (for more details [14–18]).

Also, in [19] unlike the bronze mean given in [20], a new bronze mean have been studied. A new bronze mean given in [19] can not be expressed with σ_{ρ_1,ρ_2}. Recently, a new type of manifold which is called almost poly-Norden manifold has been examined in [21]. After submanifolds of poly-Norden (semi)-Riemannian manifolds have been studied widely ([22–24]).

In this article, we studied the theory of semi-slant lightlike submanifolds of *PNsR*-manifolds.

Corresponding author: TA: tubaact@gmail.com ORCID:0000-0002-5096-3388,
Received: 15 January 2024; Accepted: 19 March 2024; Published: 30 April 2024
Keywords. Poly-Norden structure, Semi-Slant submanifold, Lightlike submanifold.
2010 *Mathematics Subject Classification.* 53C15, 53C40
Cited this article as: Acet, T. (2024). A Note on Semi-Slant Lightlike Submanifolds of PNsR-Manifolds. Turkish Journal of Science, 9(1), 32-44.

2. Preliminaries

The positive solution of $x^2 - \omega x + 1 = 0$, is named bronze mean [19], which is given by

$$\rho_\omega = \frac{\omega + \sqrt{\omega^2 - 4}}{2}. \tag{1}$$

By use of (1), B. Şahin defined a new type of manifold equipped with the bronze structure [21]. A differentiable manifold $\tilde{O}$, with a $(1,1)-$tensor field Λ and semi-Riemannian metric $\tilde{g}$ satisfying

$$\Lambda^2 = \omega\Lambda - I, \tag{2}$$

$$\tilde{g}(\Lambda\partial_1, \Lambda\partial_2) = \omega\tilde{g}(\Lambda\partial_1, \partial_2) - \tilde{g}(\partial_1, \partial_2), \tag{3}$$

then Λ is called an almost *PNsR*-manifold.

From (3), we get

$$\tilde{g}(\Lambda\partial_1, \partial_2) = \tilde{g}(\partial_1, \Lambda\partial_2),$$

for all $\partial_1, \partial_2 \in \Gamma(T\tilde{O})$.

Throught this article, we will assume that ω different from zero (see also [25]).

Definition 2.1. *[21] Let $(\tilde{O}, \tilde{g})$ be a semi-Riemannian manifold endowed with a poly-Norden structure Λ. If Λ is parallel with respect to the Levi-Civita connection $\tilde{\sharp}$, i.e.,*

$$\tilde{\sharp}\Lambda = 0, \tag{4}$$

then $(\tilde{O}, \Lambda, \tilde{g})$ is called a PNsR-manifold.

Example 2.2. *[21] Consider the 4-tuples real space $\mathbb{R}^4$ and define a map by*

$$\begin{array}{rcl} \Lambda & : & \mathbb{R}^4 \to \mathbb{R}^4 \\ (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4) & \to & (\rho_\omega\varsigma_1, \rho_\omega\varsigma_2, \bar{\rho}_\omega\varsigma_3, \bar{\rho}_\omega\varsigma_4), \end{array}$$

where $\rho_\omega = \frac{\omega+\sqrt{\omega^2-4}}{2}$ and $\bar{\rho}_\omega = \frac{\omega-\sqrt{\omega^2-4}}{2}$. Thus $(\mathbb{R}^4, \Lambda)$ is an example of almost poly-Norden manifold.

A submanifold (O^m, g) immersed in a semi-Riemannian manifold $(\tilde{O}^{m+n}, \tilde{g})$ is known a *lightlike submanifold* [1], if the metric g induced from $\tilde{g}$ is degenerate and the radical distribution $RadTO$ is of rank r, $1 \leq r \leq m$. Assume that $S(TO)$ is a screen distribution which is a semi-Riemannian complementary distribution of $RadTO$, so,

$$TO = S(TO) \perp RadTO. \tag{5}$$

Considering a screen transversal vector bundle $S(TO^\perp)$, which is a semi-Riemannian complementary vector bundle of $RadTO$ in $TO^\perp$. For every local basis $\{\zeta_i\}$ of $RadTO$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TO^\perp)$ in $(S(TO^\perp))^\perp$ such that

$$\tilde{g}(N_i, \zeta_i) = \delta_{ij} \quad \text{and} \quad \tilde{g}(N_i, N_j) = 0,$$

it follows that there exists a lightlike transversal vector bundle $ltr(TO)$ locally spanned by $\{N_i\}$ [1].

If $tr(TO)$ is a complementary (but not orthogonal) vector bundle to TO in $T\tilde{O}|_O$ then

$$tr(TO) = S(TO^\perp) \perp ltr(TO), \tag{6}$$

$$T\tilde{O}|_O = TO \oplus tr(TO), \tag{7}$$

which gives

$$T\tilde{O} = S(TO) \perp \{RadTO \oplus ltr(TO)\} \perp S(TO^\perp). \tag{8}$$

Moreover, Gauss and Weingarten formulae are given as

$$\tilde{\sharp}_{\partial_1}\partial_2 = \sharp_{\partial_1}\partial_2 + h(\partial_1, \partial_2), \tag{9}$$

$$\tilde{\sharp}_{\partial_1}N = -A_N\partial_1 + \sharp^t_{\partial_1}N, \tag{10}$$

for all $\partial_1, \partial_2 \in \Gamma(TO)$ and $N \in \Gamma(ltr(TO))$. $\sharp$ and $\sharp^t$ are linear connections on TO and $tr(TO)$, respectively.

Also, for all $\partial_1, \partial_2 \in \Gamma(TO)$ and $N \in \Gamma(ltr(TO))$ and $W \in \Gamma(S(TO^\perp))$, we get

$$\tilde{\sharp}_{\partial_1}\partial_2 = \sharp_{\partial_1}\partial_2 + h^l(\partial_1, \partial_2) + h^s(\partial_1, \partial_2), \tag{11}$$

$$\tilde{\sharp}_{\partial_1}N = -A_N\partial_1 + \sharp^l_{\partial_1}N + D^s(\partial_1, N), \tag{12}$$

$$\tilde{\sharp}_{\partial_1}W = -A_W\partial_1 + \nabla^s_{\partial_1}W + D^l(\partial_1, W). \tag{13}$$

Denote the projection of TO on $S(TO)$ by $\check{P}$. For any $\partial_1, \partial_2 \in \Gamma(TO)$ and $\zeta \in \Gamma(RadTO)$, we get

$$\sharp_{\partial_1}\check{P}\partial_2 = \sharp^*_{\partial_1}\check{P}\partial_2 + h^*(\partial_1, \check{P}\partial_2), \tag{14}$$

$$\sharp_{\partial_1}\zeta = -A^*_\zeta\partial_1 + \sharp^{*t}_{\partial_1}\zeta. \tag{15}$$

From above equations, we find

$$\tilde{g}(h^l(\partial_1, \check{P}\partial_2), \zeta) = \tilde{g}(A^*_E\partial_1, \check{P}\partial_2), \tag{16}$$

$$\tilde{g}(h^*(\partial_1, \check{P}\partial_2), N) = \tilde{g}(A_N\partial_1, \check{P}\partial_2), \tag{17}$$

$$\tilde{g}(h^l(\partial_1, \zeta), \zeta) = 0, \quad A^*_\zeta\zeta = 0. \tag{18}$$

We know that $\sharp$ is not metric connection and we have

$$(\sharp_{\partial_1}\tilde{g})(\partial_2, \partial_3) = \tilde{g}(h^l(\partial_1, \partial_2), \partial_3) + \tilde{g}(h^l(\partial_1, \partial_3), \partial_2). \tag{19}$$

3. Semi-Slant Lightlike Submanifolds of *PNsR*-Manifolds

Definition 3.1. *Let O be a lightlike submanifold of a PNsR-manifold $(\tilde{O}, \Lambda, \tilde{g})$. Then we say that O is a semi-slant lightlike submanifold if the following conditions are satisfied:*

i) $\Lambda(RadTO)$ is a distribution such that $RadTO \cap \Lambda(RadTO) = \{0\}$,

ii) There exists non-degenerate orthogonal distributions γ_1 and γ_2 on O such that

$$S(TO) = \{\Lambda(RadTO) \oplus \Lambda(ltr(TO))\} \perp \gamma_1 \perp \gamma_2,$$

iii) The distributions γ_1 is invariant, $\Lambda\gamma_1 = \gamma_1$,

iv) The distributions γ_2 is slant with angle $\phi(\neq 0)$ i.e., for each $x \in O$ and non-zero vector $X \in (\gamma_2)_x$, the angle ϕ between ΛX and the vector space $(\gamma_2)_x$ is non-zero constant, which is independent of the choice of $x \in O$ and $X \in (\gamma_2)_x$.

A semi-slant lightlike submanifold is said to be proper if $\gamma_1 \neq \{0\}$, $\gamma_2 \neq \{0\}$ and $\phi \neq \frac{\pi}{2}$.

From above definition, we arrive at

$$TO = RadTO \perp \{\Lambda RadTO \oplus \Lambda ltr(TO)\} \perp \gamma_1 \perp \gamma_2. \tag{20}$$

Example 3.2. *Let* $(\mathbb{R}_2^{12}, \tilde{g})$ *be a semi-Riemannian manifold with signature* $(-,-,+,...,+,)$ *and* $(\varsigma_1, \varsigma_2, ..., \varsigma_{12})$ *be standard coordinate system of* $\mathbb{R}_2^{12}$.

Taking

$$\Lambda(\varsigma_1, ..., \varsigma_{12}) = \left(\begin{array}{c} \bar{\rho}_\omega u_1, \rho_\omega u_2, \rho_\omega u_3, \bar{\rho}_\omega u_4, \bar{\rho}_\omega u_5, \rho_\omega u_6, \\ \bar{\rho}_\omega u_7, \rho_\omega u_8, \rho_\omega u_9, \bar{\rho}_\omega u_{10}, \bar{\rho}_\omega u_{11}, \bar{\rho}_\omega u_{12} \end{array} \right)$$

where $\rho_\omega = \frac{\omega + \sqrt{\omega^2 - 4}}{2}$ *and* $\bar{\rho}_\omega = 1 - \rho_\omega$. *Thus* Λ *is a poly-Norden structure on* $\mathbb{R}_2^{12}$.

Suppose that O *is a submanifold of* $\mathbb{R}_2^{12}$ *given by*

$$\varsigma_1 = \rho_\omega x_1 - x_2 + x_3, \quad \varsigma_2 = x_1 - \rho_\omega x_2 + \rho_\omega x_3,$$

$$\varsigma_3 = x_1 + \rho_\omega x_2 + \rho_\omega x_3, \quad \varsigma_4 = \rho_\omega x_1 + x_2 + x_3,$$

$$\varsigma_5 = \rho_\omega x_4, \quad \varsigma_6 = \rho_\omega x_5,$$

$$\varsigma_7 = \bar{\rho}_\omega x_4, \quad \varsigma_8 = \bar{\rho}_\omega x_5,$$

$$\varsigma_9 = \rho_\omega x_6, \quad \varsigma_{10} = \rho_\omega x_7,$$

$$\varsigma_{11} = \bar{\rho}_\omega x_6, \quad \varsigma_{12} = \bar{\rho}_\omega x_7.$$

Then $TO = Sp\{\Phi_1, ..., \Phi_7\}$, *where*

$$\Phi_1 = \rho_\omega \partial x_1 + \partial x_2 + \partial x_3 + \rho_\omega \partial x_4,$$

$$\Phi_2 = -\partial x_1 + \rho_\omega \partial x_2 + \rho_\omega \partial x_3 + \partial x_4,$$

$$\Phi_3 = \partial x_1 + \rho_\omega \partial x_2 + \rho_\omega \partial x_3 + \partial x_4,$$

$$\Phi_4 = \rho_\omega \partial x_5 + \bar{\rho}_\omega \partial x_7, \quad \Phi_5 = \rho_\omega \partial x_6 + \bar{\rho}_\omega \partial x_8,$$

$$\Phi_6 = \rho_\omega \partial x_9 + \bar{\rho}_\omega \partial x_{11}, \quad \Phi_7 = \rho_\omega \partial x_{10} + \bar{\rho}_\omega \partial x_{12}.$$

Thus, $RadTO = Sp\{\Phi_1\}$ *and* $S(TO) = Sp\{\Phi_2, ..., \Phi_7\}$ *and* $ltr(TO)$ *is spanned by*

$$N = \frac{1}{2(1 + \rho_\omega^2)} \left(-\rho_\omega \partial x_1 - \partial x_2 + \partial x_3 + \rho_\omega \partial x_4 \right),$$

and $S(TO^\perp)$ *is spanned by*

$$W_1 = \bar{\rho}_\omega \partial x_5 - \rho_\omega \partial x_7, \quad W_2 = \bar{\rho}_\omega \partial x_6 - \rho_\omega \partial x_8,$$

$$W_3 = \bar{\rho}_\omega \partial x_9 - \rho_\omega \partial x_{11}, \quad W_2 = \bar{\rho}_\omega \partial x_{10} - \rho_\omega \partial x_{12}.$$

It follows that $\Lambda\Phi_1 = \Phi_3$, $\Lambda N = \Phi_2$, $\Lambda\Phi_4 = \bar{\rho}_\omega \Phi_4$, $\Lambda\Phi_5 = \rho_\omega \Phi_5$ *which gives that* γ_1 *is invariant,* $\gamma_1 = Sp\{\Psi_4, \Psi_5\}$ *and* $\gamma_2 = Sp\{\Psi_6, \Psi_7\}$, *is a slant distribution. Therefore* O *is a semi-slant lightlike submanifold of* $\mathbb{R}_2^{12}$.

For any vector field $\partial_1 \in \Gamma(TO)$, we take

$$\Lambda\partial_1 = t\partial_1 + n\partial_1, \tag{21}$$

where $t\partial_1$ and $n\partial_1$ are the tangential and the transversal part of $\Lambda\partial_1$, respectively. We show the projections on $RadTG$, $\Lambda(RadTG)$, $\Lambda(ltr(TG))$, γ_1 and γ_2 by R_1, R_2, R_3, R_4 and R_5 respectively. Similarly, we show that the projections of $tr(TO)$ on $\Lambda(ltr(TO))$ and $S(TO^\perp)$ by Q_1 and Q_2, respectively. Then, we get

$$\partial_1 = R_1\partial_1 + R_2\partial_1 + R_3\partial_1 + R_4\,\partial_1 + R_5\partial_1. \tag{22}$$

Applying Λ to (22), we have

$$\begin{aligned} \Lambda\partial_1 &= \Lambda R_1\partial_1 + \Lambda R_2\partial_1 + \Lambda R_3\partial_1 \\ &\quad + \Lambda R_4\,\partial_1 + \Lambda R_5\partial_1, \end{aligned} \tag{23}$$

which gives

$$\begin{aligned}\Lambda\partial_1 &= \Lambda R_1\partial_1 + \Lambda R_2\partial_1 + \Lambda R_3\partial_1 + \Lambda R_4\,\partial_1 \\ &\quad + tR_5\partial_1 + nR_5\partial_1,\end{aligned} \tag{24}$$

where $tR_5\partial_1$ denotes the tangential component of $\Lambda R_5\partial_1$, $nR_5\partial_1$ denotes the transversal component of $\Lambda R_5\partial_1$.

Also, for any $W \in \Gamma(tr(TO))$, we have

$$W = Q_1W + Q_2W. \tag{25}$$

Applying Λ to (25), we have

$$\Lambda W = \Lambda Q_1W + \Lambda Q_2W,$$

which yields

$$\Lambda W = \Lambda Q_1W + bQ_2W + cQ_2W, \tag{26}$$

where bQ_2W denotes the tangential component of ΛQ_2W, cQ_2W denotes the transversal component of ΛQ_2W.

Thus, we obtain

$$\Lambda Q_1W \in \Gamma(\Lambda(ltr(TG)), \quad bQ_2W \in \Gamma(\gamma_2),$$
$$cQ_2W \in \Gamma(S(TO^{\perp})).$$

4. Main Results

Now, we give the main results of our article:

Theorem 4.1. *Let O be a semi-slant submanifold of a PNsR-manifold $(\tilde{O}, \Lambda, \tilde{g})$. Then $RadTO$ is integrable if and only if*

i) $\tilde{g}(h^l(\zeta_1, \Lambda\zeta_2), \zeta_3) = \tilde{g}(h^l(\zeta_2, \Lambda\zeta_1), \zeta_3)$,

ii) $\tilde{g}(h^*(\zeta_1, \Lambda\zeta_2), N) = \tilde{g}(h^*(\zeta_2, \Lambda\zeta_1), N)$,

iii) $\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, \Lambda\partial_1) = \omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, \partial_1)$,

iv) $\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, t\partial_2) + \tilde{g}(h^s(\zeta_1, \Lambda\zeta_2) - h^s(\zeta_2, \Lambda\zeta_1), n\partial_2) = \omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, \partial_1)$,

for all $\zeta_i \in \Gamma(RadTO)$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$.

Proof. It is well known that $RadTO$ is integrable iff

$$\tilde{g}([\zeta_1, \zeta_2], \Lambda\zeta_3) = \tilde{g}([\zeta_1, \zeta_2], \Lambda N) = \tilde{g}([\zeta_1, \zeta_2], \partial_1) = \tilde{g}([\zeta_1, \zeta_2], \partial_2) = 0$$

for any $\zeta_i \in \Gamma(RadTO)$, $(i = 1, 2, 3)$, $\partial_1 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$. Because of $\tilde{\sharp}$ is a metric connection, in view of (3), (11), (14) with (21), we get

$$\begin{aligned}\tilde{g}([\zeta_1, \zeta_2], \Lambda\zeta_3) &= \tilde{g}(\tilde{\sharp}_{\zeta_1}\zeta_2 - \tilde{\sharp}_{\zeta_2}\zeta_1, \Lambda\zeta_3) \\ &= \tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2 - \tilde{\sharp}_{\zeta_2}\Lambda\zeta_1, \zeta_3) \\ &= \tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2 + h^l(\zeta_1, \Lambda\zeta_2) + h^s(\zeta_1, \Lambda\zeta_2), \zeta_3) \\ &\quad - \tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1 + h^l(\zeta_2, \Lambda\zeta_1) + h^s(\zeta_2, \Lambda\zeta_1), \zeta_3) \\ &= \tilde{g}(h^l(\zeta_1, \Lambda\zeta_2), \zeta_3) - \tilde{g}(h^l(\zeta_2, \Lambda\zeta_1), \zeta_3),\end{aligned} \tag{27}$$

$$\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\Lambda N) &= \tilde{g}(\tilde{\sharp}_{\zeta_1}\zeta_2-\tilde{\sharp}_{\zeta_2}\zeta_1,\Lambda N)\\
&= \tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,N)\\
&= \tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2+h^l(\zeta_1,\Lambda\zeta_2)+h^s(\zeta_1,\Lambda\zeta_2),N)\\
&\quad -\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1+h^l(\zeta_2,\Lambda\zeta_1)+h^s(\zeta_2,\Lambda\zeta_1),N)\\
&= \tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2,N)-\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1,N),\\
&= \tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2+h^*(\zeta_1,\Lambda\zeta_2),N)\\
&\quad -\tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1+h^*(\zeta_2,\Lambda\zeta_1),N)\\
&= \tilde{g}(h^*(\zeta_1,\Lambda\zeta_2)-h^*(\zeta_2,\Lambda\zeta_1),N), \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\partial_1) &= -\tilde{g}(\Lambda[\zeta_1,\zeta_2],\Lambda\partial_1)+\omega\tilde{g}(\Lambda[\zeta_1,\zeta_2],\partial_1)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,\Lambda\partial_1)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,\partial_1)\\
&= -\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2+h^l(\zeta_1,\Lambda\zeta_2)+h^s(\zeta_1,\Lambda\zeta_2),\Lambda\partial_1)\\
&\quad +\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1+h^l(\zeta_2,\Lambda\zeta_1)+h^s(\zeta_2,\Lambda\zeta_1),\Lambda\partial_1)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2+h^l(\zeta_1,\Lambda\zeta_2)+h^s(\zeta_1,\Lambda\zeta_2),\partial_1)\\
&\quad -\omega\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1+h^l(\zeta_2,\Lambda\zeta_1)+h^s(\zeta_2,\Lambda\zeta_1),\partial_1)\\
&= -\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2,\Lambda\partial_1)+\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1,\Lambda\partial_1)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2,\partial_1)-\omega\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1,\partial_1)\\
&= -\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2+h^*(\zeta_1,\Lambda\zeta_2),\Lambda\partial_1)\\
&\quad +\tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1+h^*(\zeta_2,\Lambda\zeta_1),\Lambda\partial_1)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2+h^*(\zeta_1,\Lambda\zeta_2),\partial_1)\\
&\quad -\omega\tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1+h^*(\zeta_2,\Lambda\zeta_1),\partial_1)\\
&= \tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1-\sharp^*_{\zeta_1}\Lambda\zeta_2,\Lambda\partial_1)\\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2-\sharp^*_{\zeta_2}\Lambda\zeta_1,\partial_1), \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\zeta_1,\zeta_2],\partial_2) &= -\tilde{g}(\Lambda[\zeta_1,\zeta_2],\Lambda\partial_2)+\omega\tilde{g}(\Lambda[\zeta_1,\zeta_2],\partial_2)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,\Lambda\partial_2)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,\partial_2)\\
&= -\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\tilde{\sharp}_{\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\zeta_2}\Lambda\zeta_1,\partial_2)\\
&= -\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2+h^l(\zeta_1,\Lambda\zeta_2)+h^s(\zeta_1,\Lambda\zeta_2),t\partial_2+n\partial_2)\\
&\quad +\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1+h^l(\zeta_2,\Lambda\zeta_1)+h^s(\zeta_2,\Lambda\zeta_1),t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2+h^l(\zeta_1,\Lambda\zeta_2)+h^s(\zeta_1,\Lambda\zeta_2),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp_{\zeta_2}\Lambda\zeta_1+h^l(\zeta_2,\Lambda\zeta_1)+h^s(\zeta_2,\Lambda\zeta_1),\partial_2)
\end{aligned}$$

$$
\begin{aligned}
&= -\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2 - \sharp_{\zeta_2}\Lambda\zeta_1, t\partial_2) \\
&\quad -\tilde{g}(h^s(\zeta_1,\Lambda\zeta_2) - h^s(\zeta_2,\Lambda\zeta_1), n\partial_2) \\
&\quad +\omega\tilde{g}(\sharp_{\zeta_1}\Lambda\zeta_2 - \sharp_{\zeta_2}\Lambda\zeta_1, \partial_2) \\
&= -\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 + h^*(\zeta_1,\Lambda\zeta_2), t\partial_2) \\
&\quad +\tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1 + h^*(\zeta_2,\Lambda\zeta_1), t\partial_2) \\
&\quad -\tilde{g}(h^s(\zeta_1,\Lambda\zeta_2) - h^s(\zeta_2,\Lambda\zeta_1), n\partial_2) \\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 + h^*(\zeta_1,\Lambda\zeta_2), \partial_2) \\
&\quad -\omega\tilde{g}(\sharp^*_{\zeta_2}\Lambda\zeta_1 + h^*(\zeta_2,\Lambda\zeta_1), \partial_2) \\
&= -\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, t\partial_2) \\
&\quad -\tilde{g}(h^s(\zeta_1,\Lambda\zeta_2) - h^s(\zeta_2,\Lambda\zeta_1), n\partial_2) \\
&\quad +\omega\tilde{g}(\sharp^*_{\zeta_1}\Lambda\zeta_2 - \sharp^*_{\zeta_2}\Lambda\zeta_1, \partial_2), \qquad (30)
\end{aligned}
$$

So, we obtain the proof with equations (27)~(30). □

Theorem 4.2. *Let O be a semi-slant submanifold of a PNsR-manifold $(\tilde{O}, \Lambda, \tilde{g})$. Then $\Lambda(RadTO)$ is integrable if and only if*

i) $\tilde{g}(h^l(\Lambda\zeta_1,\zeta_2),\zeta_3) = \tilde{g}(h^l(\zeta_1,\Lambda\zeta_2),\zeta_3)$,
ii) $\tilde{g}(A^*_{\zeta_1}\Lambda\zeta_2, \Lambda\partial_1) = \tilde{g}(A^*_{\zeta_2}\Lambda\zeta_1, \Lambda\partial_1)$,
iii) $\tilde{g}(A^*_{\zeta_2}\Lambda\zeta_1 - A^*_{\zeta_1}\Lambda\zeta_2, t\partial_2) = \tilde{g}(h^s(\zeta_1,\Lambda\zeta_2) - h^s(\zeta_2,\Lambda\zeta_1), n\partial_2)$,
iv) $\tilde{g}(A_N\Lambda\zeta_1, \Lambda\zeta_2) = \tilde{g}(A_N\Lambda\zeta_2, \Lambda\zeta_1)$,
for all $\zeta_i \in \Gamma(RadTO)$, $(i = 1,2,3)$, $\partial_1 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$.

Proof. In view of the definition of semi-slant lightlike submanifold then $\Lambda(RadTO)$ is integrable iff

$$\tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\tilde{\Phi}\zeta_3) = \tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\partial_1) = \tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\partial_2) = \tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],N) = 0,$$

for any $\zeta_i \in \Gamma(RadTO)$, $(i = 1,2,3)$, $\partial_1 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$. By use of (3), (11), (12), (15) with (21) and $\sharp$ being a metric connection, we find

$$
\begin{aligned}
\tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\Lambda\zeta_3) &= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\Lambda\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\Lambda\zeta_1, \Lambda\zeta_3) \\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1), \Lambda\zeta_3) \\
&= \omega\tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1), \zeta_3) - \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1, \zeta_3) \\
&= \omega\tilde{g}((\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1), \Lambda\zeta_3) - \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1, \zeta_3) \\
&= \omega\tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2 + h^l(\Lambda\zeta_1,\zeta_2) + h^s(\Lambda\zeta_1,\zeta_2), \Lambda\zeta_3) \\
&\quad -\omega\tilde{g}(\sharp_{\Lambda\zeta_2}\zeta_1 + h^l(\Lambda\zeta_2,\zeta_1) + h^s(\Lambda\zeta_2,\zeta_1), \Lambda\zeta_3) \\
&\quad -\tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2 + h^l(\Lambda\zeta_1,\zeta_2) + h^s(\Lambda\zeta_1,\zeta_2), \zeta_3) \\
&\quad +\tilde{g}(\sharp_{\Lambda\zeta_2}\zeta_1 + h^l(\Lambda\zeta_2,\zeta_1) + h^s(\Lambda\zeta_2,\zeta_1), \zeta_3) \\
&= -\tilde{g}(h^l(\Lambda\zeta_1,\zeta_2) - h^l(\Lambda\zeta_2,\zeta_1), \zeta_3) \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\partial_1) &= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\Lambda\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\Lambda\zeta_1, \partial_1) \\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1), \partial_1) \\
&= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2 - \tilde{\sharp}_{\Lambda\zeta_2}\zeta_1, \Lambda\partial_1) \\
&= \tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2 + h^l(\Lambda\zeta_1,\zeta_2) + h^s(\Lambda\zeta_1,\zeta_2), \Lambda\partial_1) \\
&\quad -\tilde{g}(\sharp_{\Lambda\zeta_2}\zeta_1 + h^l(\Lambda\zeta_2,\zeta_1) + h^s(\Lambda\zeta_2,\zeta_1), \Lambda\partial_1) \\
&= \tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2, \Lambda\partial_1) - \tilde{g}(\sharp_{\Lambda\zeta_2}\zeta_1, \Lambda\partial_1) \\
&= \tilde{g}(-A^*_{\zeta_2}\Lambda\zeta_1 + \sharp^{*t}_{\Lambda\zeta_1}\zeta_2, \Lambda\partial_1) - \tilde{g}(-A^*_{\zeta_1}\Lambda\zeta_2 + \sharp^{*t}_{\Lambda\zeta_2}\zeta_1, \Lambda\partial_1) \\
&= (A^*_{\zeta_1}\Lambda\zeta_2 - A^*_{\zeta_2}\Lambda\zeta_1, \Lambda\partial_1), \qquad (32)
\end{aligned}
$$

$$\begin{aligned}
\tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],\partial_2) &= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\Lambda\zeta_2}\Lambda\zeta_1,\partial_2)\\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2-\tilde{\sharp}_{\Lambda\zeta_2}\zeta_1),\partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2-\tilde{\sharp}_{\Lambda\zeta_2}\zeta_1,\Lambda\partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\zeta_2-\tilde{\sharp}_{\Lambda\zeta_2}\zeta_1,t\partial_2+n\partial_2)\\
&= \tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2+h^l(\Lambda\zeta_1,\zeta_2)+h^s(\Lambda\zeta_1,\zeta_2),t\partial_2+n\partial_2)\\
&\quad -\tilde{g}(\sharp_{\Lambda\zeta_2}\zeta_1+h^l(\Lambda\zeta_2,\zeta_1)+h^s(\Lambda\zeta_2,\zeta_1),t\partial_2+n\partial_2)\\
&= \tilde{g}(\sharp_{\Lambda\zeta_1}\zeta_2-\sharp_{\Lambda\zeta_2}\zeta_1,t\partial_2)\\
&\quad +\tilde{g}(h^s(\Lambda\zeta_1,\zeta_2)-h^s(\Lambda\zeta_2,\zeta_1),n\partial_2)\\
&= \tilde{g}(-A^*_{\zeta_2}\Lambda\zeta_1+\sharp^{*t}_{\Lambda\zeta_1}\zeta_2,t\partial_2)-\tilde{g}(-A^*_{\zeta_1}\Lambda\zeta_2+\sharp^{*t}_{\Lambda\zeta_2}\zeta_1,t\partial_2)\\
&\quad +\tilde{g}(h^s(\Lambda\zeta_1,\zeta_2)-h^s(\Lambda\zeta_2,\zeta_1),n\partial_2)\\
&= \tilde{g}(A^*_{\zeta_1}\Lambda\zeta_2-A^*_{\zeta_2}\Lambda\zeta_1,t\partial_2) \qquad (33)\\
&\quad +\tilde{g}(h^s(\Lambda\zeta_1,\zeta_2)-h^s(\Lambda\zeta_2,\zeta_1),n\partial_2),
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Lambda\zeta_1,\Lambda\zeta_2],N) &= \tilde{g}(\tilde{\sharp}_{\Lambda\zeta_1}\Lambda\zeta_2-\tilde{\sharp}_{\Lambda\zeta_2}\Lambda\zeta_1,N)\\
&= -\tilde{g}(\Lambda\zeta_2,\tilde{\sharp}_{\Lambda\zeta_1}N)+\tilde{g}(\Lambda\zeta_1,\tilde{\sharp}_{\Lambda\zeta_2}N)\\
&= -\tilde{g}(-A_N\Lambda\zeta_1+\sharp^l_{\Lambda\zeta_1}N+D^s(\Lambda\zeta_1,N),\Lambda\zeta_2)\\
&\quad +\tilde{g}(-A_N\Lambda\zeta_2+\sharp^l_{\Lambda\zeta_2}N+D^s(\Lambda\zeta_2,N),\Lambda\zeta_1)\\
&= \tilde{g}(A_N\Lambda\zeta_1,\Lambda\zeta_2)-\tilde{g}(A_N\Lambda\zeta_2,\Lambda\zeta_1). \qquad (34)
\end{aligned}$$

So, proof obtaines from (31)∼(34). □

Theorem 4.3. *Let O be a bi-slant submanifold of a PNsR-manifold $(\tilde{O},\Lambda,\tilde{g})$. Then $\Lambda(ltr(TO))$ is integrable if and only if*

i) $\tilde{g}(A_{N_1}\Lambda N_2,N_3)=\tilde{g}(A_{N_2}\Lambda N_1,N_3)$,
ii) $\tilde{g}(A_{N_1}\Lambda N_2,\Lambda\partial_1)=\tilde{g}(A_{N_2}\Lambda N_1,\Lambda\partial_1)$,
iii) $\tilde{g}(A_{N_1}\Lambda N_2-A_{N_2}\Lambda N_1,t\partial_2)=\tilde{g}(D^s(\Lambda N_2,N_1)-D^s(\Lambda N_1,N_2),n\partial_2)$,
iv) $\tilde{g}(A_{N_3}\Lambda N_1,\Lambda N_2)=\tilde{g}(A_{N_3}\Lambda N_2,\Lambda N_1)$,
for all $N_i\in\Gamma(\Lambda(ltr(TO)))$, $(i=1,2,3)$, $\partial_1\in\Gamma(\gamma_1)$ and $\partial_2\in\Gamma(\gamma_2)$.

Proof. We know that $\Lambda(ltr(TO))$ is integrable iff

$$\tilde{g}([\Lambda N_1,\Lambda N_2],\Lambda N_3)=\tilde{g}([\Lambda N_1,\Lambda N_2],\partial_1)=\tilde{g}([\Lambda N_1,\Lambda N_2],\partial_2)=\tilde{g}([\Lambda N_1,\Lambda N_2],N_3)=0,$$

for any $N_i\in\Gamma(\Lambda(ltr(TO)))$, $(i=1,2,3)$, $\partial_1\in\Gamma(\gamma_1)$ and $\partial_2\in\Gamma(\gamma_2)$. In view of (3), (11), (12), (15) with (21) and $\tilde{\sharp}$ being a metric connection, we get

$$\begin{aligned}
\tilde{g}([\Lambda N_1,\Lambda N_2],\Lambda N_3) &= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}\Lambda N_2-\tilde{\sharp}_{\Lambda N_2}\Lambda N_1,\Lambda N_3)\\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda N_1}N_2-\tilde{\sharp}_{\Lambda N_2}N_1),\Lambda N_3)\\
&= \omega\tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda N_1}N_2-\tilde{\sharp}_{\Lambda N_2}N_1),N_3)-\tilde{g}(\tilde{\sharp}_{\Lambda N_1}N_2-\tilde{\sharp}_{\Lambda N_2}N_1,N_3)\\
&= \omega\tilde{g}((\tilde{\sharp}_{\Lambda N_1}N_2-\tilde{\sharp}_{\Lambda N_2}N_1),\Lambda N_3)-\tilde{g}(\tilde{\sharp}_{\Lambda N_1}N_2-\tilde{\sharp}_{\Lambda N_2}N_1,N_3)\\
&= \omega\tilde{g}(-A_{N_2}\Lambda N_1+\sharp^l_{\Lambda N_1}N_2+D^s(\Lambda N_1,N_2),\Lambda N_3)\\
&\quad -\omega\tilde{g}(-A_{N_1}\Lambda N_2+\sharp^l_{\Lambda N_2}N_1+D^s(\Lambda N_2,N_1),\Lambda N_3)\\
&\quad -\tilde{g}(-A_{N_2}\Lambda N_1+\sharp^l_{\Lambda N_1}N_2+D^s(\Lambda N_1,N_2),N_3)\\
&\quad +\tilde{g}(-A_{N_1}\Lambda N_2+\sharp^l_{\Lambda N_2}N_1+D^s(\Lambda N_2,N_1),N_3)\\
&= \tilde{g}(A_{N_2}\Lambda N_1-A_{N_1}\Lambda N_2,N_3), \qquad (35)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Lambda N_1, \Lambda N_2], \partial_1) &= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}\Lambda N_2 - \tilde{\sharp}_{\Lambda N_2}\Lambda N_1, \partial_1)\\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda N_1}N_2 - \tilde{\sharp}_{\Lambda N_2}N_1), \partial_1)\\
&= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}N_2 - \tilde{\sharp}_{\Lambda N_2}N_1, \Lambda\partial_1)\\
&= \tilde{g}(-A_{N_2}\Lambda N_1 + \sharp^l_{\Lambda N_1}N_2 + D^s(\Lambda N_1, N_2), \Lambda\partial_1)\\
&\quad -\tilde{g}(-A_{N_1}\Lambda N_2 + \sharp^l_{\Lambda N_2}N_1 + D^s(\Lambda N_2, N_1), \Lambda\partial_1)\\
&= \tilde{g}(A_{N_1}\Lambda N_2 - A_{N_2}\Lambda N_1, \Lambda\partial_1), \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Lambda N_1, \Lambda N_2], \partial_2) &= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}\Lambda N_2 - \tilde{\sharp}_{\Lambda N_2}\Lambda N_1, \partial_2)\\
&= \tilde{g}(\Lambda(\tilde{\sharp}_{\Lambda N_1}N_2 - \tilde{\sharp}_{\Lambda N_2}N_1), \partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}N_2 - \tilde{\sharp}_{\Lambda N_2}N_1, \Lambda\partial_2)\\
&= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}N_2 - \tilde{\sharp}_{\Lambda N_2}N_1, t\partial_2 + n\partial_2)\\
&= \tilde{g}(-A_{N_2}\Lambda N_1 + \sharp^l_{\Lambda N_1}N_2 + D^s(\Lambda N_1, N_2), t\partial_2 + n\partial_2)\\
&\quad -\tilde{g}(-A_{N_1}\Lambda N_2 + \sharp^l_{\Lambda N_2}N_1 + D^s(\Lambda N_2, N_1), t\partial_2 + n\partial_2)\\
&= \tilde{g}(A_{N_1}\Lambda N_2 - A_{N_2}\Lambda N_1, t\partial_2)\\
&\quad +\tilde{g}(D^s(\Lambda N_1, N_2) - D^s(\Lambda N_2, N_1), n\partial_2), \qquad (37)
\end{aligned}$$

$$\begin{aligned}
\tilde{g}([\Lambda N_1, \Lambda N_2], N_3) &= \tilde{g}(\tilde{\sharp}_{\Lambda N_1}\Lambda N_2 - \tilde{\sharp}_{\Lambda N_2}\Lambda N_1, N_3)\\
&= -\tilde{g}(\Lambda N_2, \tilde{\sharp}_{\Lambda N_1}N_3) + \tilde{g}(\Lambda N_1, \tilde{\sharp}_{\Lambda N_2}N_3)\\
&= -\tilde{g}(-A_N\Lambda N_1 + \sharp^l_{\Lambda N_1}N + D^s(\Lambda N_1, N_3), \Lambda N_2)\\
&\quad +\tilde{g}(-A_N\Lambda N_2 + \sharp^l_{\Lambda\zeta_2}N + D^s(\Lambda N_2, N_3), \Lambda N_1)\\
&= \tilde{g}(A_{N_3}\Lambda N_1, \Lambda N_2) - \tilde{g}(A_{N_3}\Lambda N_2, \Lambda N_1). \qquad (38)
\end{aligned}$$

The proof follows from (35)~(38). □

Theorem 4.4. *Let O be a semi-slant submanifold of a PNsR-manifold $(\tilde{O}, \Lambda, \tilde{g})$. Then γ is integrable if and only if*

i) $\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1 - \sharp^*_{\partial_1}\Lambda\partial_4, t\partial_2) + \tilde{g}(h^s(\partial_4, \Lambda\partial_1) - h^s(\partial_1, \Lambda\partial_4), n\partial_2) = \omega\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1 - \sharp^*_{\partial_1}\Lambda\partial_4, \partial_2)$,

ii) $\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1 - \sharp^*_{\partial_1}\Lambda\partial_4, \Lambda N) = \tilde{g}(h^*(\partial_4, \Lambda\partial_1) - h^*(\partial_1, \Lambda\partial_4), N)$,

iii) $\tilde{g}(A_N\partial_4, \Lambda\partial_1) = \tilde{g}(A_N\partial_1, \Lambda\partial_4)$,

for all $\partial_1, \partial_4 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$.

Proof. If we consider the definition of the semi-slant lightlike submanifolds then γ_1 is integrable iff

$$\tilde{g}([\partial_4, \partial_1], \partial_2) = \tilde{g}([\partial_4, \partial_1], N) = \tilde{g}([\partial_4, \partial_1], \Lambda N) = 0,$$

for any $\partial_1, \partial_4 \in \Gamma(\gamma_1)$, $\partial_2 \in \Gamma(\gamma_2)$ and $N \in \Gamma(ltr(TO))$. By use of (3), (11), (12), (14) with (21) and $\tilde{\sharp}$ being a metric connection, we get

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],\partial_2) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4,\partial_2)\\
&= -\tilde{g}(\Lambda(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),\Lambda\partial_2)+\omega\tilde{g}(\Lambda(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1+h^l(\partial_4,\Lambda\partial_1)+h^s(\partial_4,\Lambda\partial_1),t\partial_2+n\partial_2)\\
&\quad +\tilde{g}(\sharp_{\partial_1}\Lambda\partial_4+h^l(\partial_1,\Lambda\partial_4)+h^s(\partial_1,\Lambda\partial_4),t\partial_2+n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1+h^l(\partial_4,\Lambda\partial_1)+h^s(\partial_4,\Lambda\partial_1),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp_{\partial_1}\Lambda\partial_4+h^l(\partial_1,\Lambda\partial_4)+h^s(\partial_1,\Lambda\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1-\sharp_{\partial_1}\Lambda\partial_4,t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\Lambda\partial_1)-h^s(\partial_1,\Lambda\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1-\sharp_{\partial_1}\Lambda\partial_4,\partial_2)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1+h^*(\partial_4,\Lambda\partial_1),t\partial_2)\\
&\quad +\tilde{g}(\sharp^*_{\partial_1}\Lambda\partial_4+h^*(\partial_1,\Lambda\partial_4),t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\Lambda\partial_1)-h^s(\partial_1,\Lambda\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1+h^*(\partial_4,\Lambda\partial_1),\partial_2)\\
&\quad -\omega\tilde{g}(\sharp^*_{\partial_1}\Lambda\partial_4+h^*(\partial_1,\Lambda\partial_4),\partial_2)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1-\sharp^*_{\partial_1}\Lambda\partial_4,t\partial_2)\\
&\quad -\tilde{g}(h^s(\partial_4,\Lambda\partial_1)-h^s(\partial_1,\Lambda\partial_4),n\partial_2)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1+\sharp^*_{\partial_1}\Lambda\partial_4,\partial_2), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4,N)\\
&= -\tilde{g}(\Lambda(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),\Lambda N)+\omega\tilde{g}(\Lambda(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4),N)\\
&= -\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1+h^l(\partial_4,\Lambda\partial_1)+h^s(\partial_4,\Lambda\partial_1),\Lambda N)\\
&\quad +\tilde{g}(\sharp_{\partial_1}\Lambda\partial_4+h^l(\partial_1,\Lambda\partial_4)+h^s(\partial_1,\Lambda\partial_4),\Lambda N)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1+h^l(\partial_4,\Lambda\partial_1)+h^s(\partial_4,\Lambda\partial_1),N)\\
&\quad -\omega\tilde{g}(\sharp_{\partial_1}\Lambda\partial_4+h^l(\partial_1,\Lambda\partial_4)+h^s(\partial_1,\Lambda\partial_4),N)\\
&= -\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1-\sharp_{\partial_1}\Lambda\partial_4,\Lambda N)+\omega\tilde{g}(\sharp_{\partial_4}\Lambda\partial_1-\sharp_{\partial_1}\Lambda\partial_4,N)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1+h^*(\partial_4,\Lambda\partial_1),\Lambda N)\\
&\quad +\tilde{g}(\sharp^*_{\partial_1}\Lambda\partial_4+h^*(\partial_1,\Lambda\partial_4),\Lambda N)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1+h^*(\partial_4,\Lambda\partial_1),N)\\
&\quad -\omega\tilde{g}(\sharp^*_{\partial_1}\Lambda\partial_4+h^*(\partial_1,\Lambda\partial_4),N)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1-\sharp^*_{\partial_1}\Lambda\partial_4,\Lambda N) \qquad (40)\\
&\quad +\omega\tilde{g}(h^*(\partial_4,\Lambda\partial_1)+h^*(\partial_1,\Lambda\partial_4),N),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}([\partial_4,\partial_1],\Lambda N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1-\tilde{\sharp}_{\partial_1}\partial_4,\Lambda N)\\
&= -\tilde{g}(\Lambda\partial_1,\tilde{\sharp}_{\partial_4}N)+\tilde{g}(\Lambda\partial_4,\tilde{\sharp}_{\partial_1}N)\\
&= -\tilde{g}(-A_N\partial_4+\sharp^l_{\partial_4}N+D^s(\partial_4,N),\Lambda\partial_1)\\
&\quad +\tilde{g}(-A_N\partial_1+\sharp^l_{\partial_1}N+D^s(\partial_1,N),\Lambda\partial_4)\\
&= \tilde{g}(A_N\partial_4,\Lambda\partial_1)-\tilde{g}(A_N\partial_1,\Lambda\partial_4). \qquad (41)
\end{aligned}
$$

So, we arrive at the proof from (39)~(41). □

Now, we obtain the necessary and sufficient conditions for a foliation determined by distribution on a semi-slant lightlike submanifolds of a *PNsR*-manifold to be totally geodesic.

Theorem 4.5. *Let O be a semi-slant submanifold of a PNsR-manifold* $(\tilde{O}, \Lambda, \tilde{g})$*. Then* γ_1 *defines totally geodesic foliation if and only if*

i) $\tilde{g}(\sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4, \Lambda\partial_1) = \omega\tilde{g}(\sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4, \partial_1)$,

ii) $\tilde{g}(\sharp^*_{\partial_4} \Lambda\partial_1, \Lambda N) = \omega\tilde{g}(h^*(\partial_4, \Lambda\partial_1), N)$,

iii) $h^*(\partial_4, \Lambda\partial_1)$ *has no component in* $\Gamma(RadTO)$,

for all $\partial_1, \partial_4 \in \Gamma(\gamma_1)$*,* $\partial_2 \in \Gamma(\gamma_2)$ *and* $N \in \Gamma(ltr(TO))$*.*

Proof. The distribution γ_1 defines totally geodesic foliation iff $\sharp_{\partial_4}\partial_1 \in \Gamma(\gamma_1)$ for all $\partial_1, \partial_4 \in \Gamma(\gamma_1)$. $\tilde{\sharp}$ being a metric connection and from (3), (11), (13) and (21), we have

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, \partial_2) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1, \partial_2)\\
&= -\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}\partial_2)\\
&= \tilde{g}(\Lambda\partial_1, \tilde{\sharp}_{\partial_4}(t\partial_2 + n\partial_2)) - \omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4}(t\partial_2 + n\partial_2))\\
&= \tilde{g}(\Lambda\partial_1, \tilde{\sharp}_{\partial_4} t\partial_2) + \tilde{g}(\Lambda\partial_1, \tilde{\sharp}_{\partial_4} n\partial_2)\\
&\quad -\omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4} t\partial_2) - \omega\tilde{g}(\partial_1, \tilde{\sharp}_{\partial_4} n\partial_2)\\
&= \tilde{g}(\Lambda\partial_1, \sharp_{\partial_4} t\partial_2 + h^l(\partial_4, t\partial_2) + h^s(\partial_4, t\partial_2))\\
&\quad +\tilde{g}(\Lambda\partial_1, -A_{n\partial_2}\partial_4 + \sharp^l_{\partial_4} n\partial_2 + D^s(\partial_4, n\partial_2))\\
&\quad -\omega\tilde{g}(\partial_1, \sharp_{\partial_4} t\partial_2 + h^l(\partial_4, t\partial_2) + h^s(\partial_4, t\partial_2))\\
&\quad -\omega\tilde{g}(\partial_1, -A_{n\partial_2}\partial_4 + \sharp^l_{\partial_4} n\partial_2 + D^s(\partial_4, n\partial_2))\\
&= \tilde{g}(\Lambda\partial_1, \sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4) - \omega\tilde{g}(\partial_1, \sharp_{\partial_4} t\partial_2 - A_{n\partial_2}\partial_4).
\end{aligned}$$

Similarly, from (3), (11) and (14), we have

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\partial_1, N)\\
&= -\tilde{g}(\tilde{\sharp}_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) + \omega\tilde{g}(\tilde{\sharp}_{\partial_4}\tilde{\Phi}\partial_1, N)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4, \tilde{\Phi}\partial_1) + h^s(\partial_4, \tilde{\Phi}\partial_1), \tilde{\Phi}N)\\
&\quad +\omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1 + h^l(\partial_4, \tilde{\Phi}\partial_1) + h^s(\partial_4, \tilde{\Phi}\partial_1), N)\\
&= -\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) + \omega\tilde{g}(\sharp_{\partial_4}\tilde{\Phi}\partial_1, N)\\
&= -\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4, \tilde{\Phi}\partial_1), \tilde{\Phi}N)\\
&\quad +\omega\tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1 + h^*(\partial_4, \tilde{\Phi}\partial_1), N)\\
&= \tilde{g}(\sharp^*_{\partial_4}\tilde{\Phi}\partial_1, \tilde{\Phi}N) - \omega\tilde{g}(h^*(\partial_4, \tilde{\Phi}\partial_1), N).
\end{aligned}$$

Furthermore, using (3), (11) and (14), we obtain

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_4}\partial_1, \Lambda N) &= \tilde{g}(\tilde{\sharp}_{\partial_4}\Lambda\partial_1, N)\\
&= \tilde{g}(\sharp_{\partial_4}\Lambda\partial_1 + h^l(\partial_4, \Lambda\partial_1) + h^s(\partial_4, \Lambda\partial_1), N)\\
&= \tilde{g}(\sharp^*_{\partial_4}\Lambda\partial_1 + h^*(\partial_4, \Lambda\partial_1), N)\\
&= \tilde{g}(h^*(\partial_4, \Lambda\partial_1), N).
\end{aligned}$$

So, the proof is completed. □

Theorem 4.6. *Let O be a semi-slant submanifold of a PNsR-manifold* $(\tilde{O}, \Lambda, \tilde{g})$*. Then* γ_2 *defines totally geodesic foliation if and only if*

i) $\tilde{g}(t\partial_3, \sharp_{\partial_2}\Lambda\partial_1) + \tilde{g}(n\partial_3, h^s(\partial_2, \Lambda\partial_1)) = \omega\tilde{g}(\sharp_{\partial_2}\Lambda\partial_1, \partial_3)$,
ii) $\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, \Lambda N) = \omega\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N)$,
iii) $\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2$ *has no component in* $\Gamma(RadTO)$,
for all $\partial_1 \in \Gamma(\gamma_1)$, $\partial_2, \partial_3 \in \Gamma(\gamma_2)$ *and* $N \in \Gamma(ltr(TO))$.

Proof. The distribution γ_2 defines totally geodesic foliation iff $\sharp_{\partial_2}\partial_3 \in \Gamma(\gamma_2)$ for all $\partial_2, \partial_3 \in \Gamma(\gamma_2)$. In view of (3), (11) and (21) with the properties of the connection $\tilde{\sharp}$, we get

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, \partial_1) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, \partial_1) \\
&= -\tilde{g}(\partial_3, \tilde{\sharp}_{\partial_2}\partial_1) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}\Lambda\partial_1, \Lambda\partial_3) - \omega\tilde{g}(\tilde{\sharp}_{\partial_2}\Lambda\partial_1, \partial_3) \\
&= \tilde{g}(\sharp_{\partial_2}\Lambda\partial_1 + h^l(\partial_2, \Lambda\partial_1) + h^s(\partial_2, \Lambda\partial_1), t\partial_3 + n\partial_3) \\
&\quad -\omega\tilde{g}(\sharp_{\partial_2}\Lambda\partial_1 + h^l(\partial_2, \Lambda\partial_1) + h^s(\partial_2, \Lambda\partial_1), \partial_3) \\
&= \tilde{g}(\sharp_{\partial_2}\Lambda\partial_1, t\partial_3) + \tilde{g}(h^s(\partial_2, \Lambda\partial_1), n\partial_3) \\
&\quad -\omega\tilde{g}(\sharp_{\partial_2}\Lambda\partial_1, \partial_3).
\end{aligned}$$

Similarly, from (3), (11), (13) and (21), we have

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}\Lambda\partial_3, \Lambda N) + \omega\tilde{g}(\tilde{\sharp}_{\partial_2}\Lambda\partial_3, N) \\
&= -\tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), \Lambda N) + \omega\tilde{g}((t\partial_3 + n\partial_3), N) \\
&= -\tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), \Lambda N) \\
&\quad -\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), \Lambda N) \\
&\quad +\omega\tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), N) \\
&\quad +\omega\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), N) \\
&= -\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, \Lambda N) + \omega\tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N).
\end{aligned}$$

Also, from (3), (11), (13) and (21), we get

$$\begin{aligned}
\tilde{g}(\sharp_{\partial_2}\partial_3, N) &= \tilde{g}(\tilde{\sharp}_{\partial_2}\partial_3, \Lambda N) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}\Lambda\partial_3, N) \\
&= \tilde{g}(\tilde{\sharp}_{\partial_2}(t\partial_3 + n\partial_3), N) \\
&= \tilde{g}(\sharp_{\partial_2}t\partial_3 + h^l(\partial_2, t\partial_3) + h^s(\partial_2, t\partial_3), N) \\
&\quad +\tilde{g}(-A_{n\partial_3}\partial_2 + \sharp^l_{\partial_2}n\partial_3 + D^s(\partial_2, n\partial_3), N) \\
&= \tilde{g}(\sharp_{\partial_2}t\partial_3 - A_{n\partial_3}\partial_2, N).
\end{aligned}$$

which gives proof of our assertion. □

References

[1] Duggal, K.L., Bejancu, A. (1996). Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and Its Applications. Kluwer Publisher.
[2] Duggal, K.L., Şahin, B. (2010). Differential geometry of lightlike submanifolds, Frontiers in Mathematics.
[3] Duggal, K.L., Şahin, B. (2006). Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds, Acta Math Hungar., 112, 107 - 130.
[4] Şahin, B. (2008). Slant lightlike submanifolds of indefinite Hermitian manifolds, Balkan J Geo Appl., 13, 107 - 119.
[5] Erdoğan, F.E., Yüksel Perktaş, Bozdağ, Ş.N., Acet, B.E. (2023). Lightlike hypersurfaces of meta golden semi-Riemannian manifolds, Mathematics, 11, 1 - 16.

[6] Chen, B.-Y. (1990). Geometry of slant submanifolds, Katholike Universiteit Leuven, Louvain, 123pp.
[7] Papaghiuc, N. (1994). Semi-slant submanifolds of a Kaehlerian manifold, An. Stiint. Univ. Al. I. Cuza Iasi Sect. I a Mat., 40(1), 55 - 61.
[8] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., Fernandez, M. (2000). Slant submanifolds in Sasakian manifolds. Glasg. Math. J., 42(1), 125 - 138.
[9] Alegre, P., Carriazo, A. (2019). Bi-slant submanifolds of para-Hermitian manifolds, Mathematics, 7.
[10] Ahmad, M., Ahmad, M., Mofarreh, F. (2023). Bi-slant lightlike submanifolds of golden semi-Riemannian manifolds, Mathematics, 8, 19526 - 19545.
[11] Şahin, B. (2006). Slant submanifolds of almost product Riemannian manifolds, J Korean Math Soc., 43, 717 - 732.
[12] Spinadel, V.W. (2002). The metallic means family and forbidden symmetries, Int Math J., 2(3), 279 - 288.
[13] Hretcanu, C.E., Crasmareanu, M.C. (2013). Metallic structure on Riemannian manifolds, Rev Un Mat Argentina, 54(2), 15 - 27.
[14] Acet, B.E. (2018). Lightlike hypersurfaces of metallic semi-Riemannian manifolds, Int J Geo Meth Modern Phys., 15(12), 185 – 201.
[15] Blaga, A.M., Hretcanu, C.E. (2018). Invariant, anti-invariant and slant submanifolds of metallic Riemannian manifolds, Novi Sad J Math., 48(2), 55 – 80.
[16] Hretcanu, C.E., Blaga, A.M. (2018). Submanifolds in metallic semi-Riemannian manifolds, Differ Geom Dynm Syst., 20, 83 - 97.
[17] Yüksel Perktaş, S., Erdoğan, F.E., Acet, B.E., (2020). Lightlike submanifolds of metallic semi-Riemannian manifolds, Filomat, 34(6), 1781 - 1794.
[18] Erdoğan, F.E., Yüksel Perktaş, S., Acet, B.E., Blaga, A.M. (2019). Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds, J Geom Phys., 142, 111 - 120.
[19] Kalia, S. The generalizations of the golden ratio, their powers, continued fractions and convergents, http://math.mit.edu/research/highschool/primes/papers.php.
[20] Spinadel, V.W. (1999). The metallic means family and multifractal spectra, Nonlinear Analy Ser B: Real World Appl., 36(6), 721 - 745.
[21] Şahin, B. (2018). Almost poly-Norden manifolds, Int J Maps in Math., 1(1), 68 - 79.
[22] Yüksel Perktaş, S. (2020). Submanifolds of poly-Norden Riemannian manifolds, Turk J Math., 44, 31 - 49.
[23] Kılıç, E., Acet, T., Yüksel Perktaş, S. (2022). Lightlike hypersurfaces of poly-Norden semi-Riemannian manifolds, Turk J Sci., 7(1), 21 - 30.
[24] Yüksel Perktaş, S., Acet, T., Kılıç, E. (2023). On lightlike submanifolds of poly-Norden semi-Riemannian manifolds, Turk J Math Comp Sci., 15(1), 1 - 11.
[25] Acet, T., Yüksel Perktaş, S., Kılıç, E. (2023). On some types of lightlike submanifolds of poly-Norden semi-Riemannian manifolds, Filomat, 37(10), 3725 - 3740.